



Construction of a global solution for the one dimensional singularly-perturbed boundary value problem

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Abstract

We consider an approximate solution for the one-dimensional semilinear singularly-perturbed boundary value problem, using the previously obtained numerical values of the boundary value problem in the mesh points and the representation of the exact solution using Green's function. We present an ε -uniform convergence of such gained the approximate solutions, in the maximum norm of the order $O(N^{-1})$ on the observed domain. After that, the constructed approximate solution is repaired and we obtain a solution, which also has ε -uniform convergence, but now of order $O(\ln^2 N/N^2)$ on $[0, 1]$. In the end a numerical experiment is presented to confirm previously shown theoretical results.

Keywords: Singular perturbation, nonlinear, boundary layer, Shishkin mesh, layer-adapted mesh, uniform convergence.

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1. Introduction

We will consider the singularly-perturbed boundary value problem

$$\varepsilon^2 y'' = f(x, y), \quad x \in I = [0, 1], \quad (1)$$

$$y(0) = 0, \quad y(1) = 0, \quad (2)$$

with the condition

$$\frac{\partial f(x, y)}{\partial y} := f_y \geq m > 0, \quad \forall (x, y) \in I \times \mathbb{R}, \quad (3)$$

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where $0 < \varepsilon < 1$ is a perturbation parameter, f is a nonlinear function $f \in C^k(I \times \mathbb{R})$, $k \geq 2$ and m is a real constant.

The boundary value problem (1)–(2), with the condition (3), has a unique solution, see [16]. Contributions to numerical solutions of the problem (1)–(2) with different assumptions on the function f and similar problems were obtained by many authors, see for example Flaherty and O'Malley [6], Cvetković and Herceg [2], Herceg [7, 8], Herceg, Surla and Rapajić [9], Kopteva [11], Linß and Vulcanović [15], Nijjima [17], Stynes and O'Riordan [18], Vulcanović [19, 21, 22, 23] etc.

The method that will be used in this paper in order to obtain a discrete approximate solution, i.e. values of the approximate solution in the mesh points, of the problem (1)–(3) was first developed by Boglaev [1], who constructed a difference scheme and showed convergence of order 1 on the modified Bakhvalov mesh. Using the method of [1], we constructed new difference schemes in [3, 4] and we carried out numerical experiments.

In [5, 10] we constructed new difference schemes and we proved uniqueness of the numerical solution and ε -uniform convergence on the modified Shishkin mesh and at the end presented numerical experiments. In this paper we will use the difference scheme from [10] in order to calculate values of the approximate solution of the problem on the mesh points and then construct an approximate solution.

2. Theoretical background and known results

Let us set up an arbitrary mesh on $[0, 1]$

$$0 = x_0 < x_1 < \dots < x_N = 1. \quad (4)$$

A construction of a difference scheme, which will be used for calculation of the approximate solution of the problem (1)–(3) in the mesh points, is based on the representation of the exact solution on the interval $[x_i, x_{i+1}]$, $i = 0, \dots, N - 1$

$$y_i(x) = y_i u_i^I(x) + y_{i+1} u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x, s) \psi(s, y(s)) \, ds, \quad (5)$$

where $G_i(x, s)$ is the Green's function

$$G_i(x, s) = \frac{1}{\varepsilon^2 w_i(s)} \begin{cases} u_i^{II}(x) u_i^I(s), & x_i \leq x \leq s \leq x_{i+1}, \\ u_i^I(x) u_i^{II}(s), & x_i \leq s \leq x \leq x_{i+1}, \end{cases} \quad (6)$$

$$\psi(s, y(s)) = f(s, y(s)) - \gamma y(s), \quad (7)$$

and $w_i(s) = \frac{-\beta}{\sinh(\beta h_i)}$, $s \in [x_i, x_{i+1}]$, $u_i^I(x) = \frac{\sinh(\beta(x_{i+1}-x))}{\sinh(\beta h_i)}$, $u_i^{II}(x) = \frac{\sinh(\beta(x-x_i))}{\sinh(\beta h_i)}$, $h_i = x_{i+1} - x_i$, $\beta = \frac{\sqrt{\gamma}}{\varepsilon}$, $y_i := y(x_i)$ and γ is a constant for which $\gamma \geq f_y$, (details can be found in [10]). The difference scheme constructed in [10], which we will use, has the following form

$$\frac{a_i + d_i}{2} \bar{y}_{i-1} - \left(\frac{a_i + d_i}{2} + \frac{a_{i+1} + d_{i+1}}{2} \right) \bar{y}_i + \frac{a_{i+1} + d_{i+1}}{2} \bar{y}_{i+1} = \frac{\Delta d_i}{\gamma} \bar{f}_{i-1} + \frac{\Delta d_{i+1}}{\gamma} \bar{f}_i, \quad (8)$$

where \bar{y}_k , $k \in \{i-1, i, i+1\}$ are values of the approximate solution in the mesh points, $\Delta d_i = d_i - a_i$, $d_i = \frac{1}{\tanh(\beta h_i)}$, $a_i = \frac{1}{\sinh(\beta h_i)}$ and $\bar{f}_i = f((x_i + x_{i+1})/2, (\bar{y}_i + \bar{y}_{i+1})/2)$, $i = 1, \dots, N - 1$. The difference scheme generates a system of nonlinear equations and the solutions of this system are values of the approximate solution in the mesh points. An answer to the question of existence and uniqueness will be given in the next theorem, however before that, it is necessary to define the operator (or discrete problem) $F : \mathbb{R}^{N+1} \mapsto \mathbb{R}^{N+1}$

and a corresponding norm that is necessary in formulation of the theorem. Therefore, we will now use the difference scheme (8) in order to obtain a discrete problem of the problem (1)–(3). We have that

$$F\bar{y} = \left((F\bar{y})_0, (F\bar{y})_1, \dots, (F\bar{y})_N \right)^T = 0, \quad (9)$$

where

$$\begin{aligned} (F\bar{y})_0 &:= 0, \\ (F\bar{y})_i &:= \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[\frac{a_i + d_i}{2} \bar{y}_{i-1} - \left(\frac{a_i + d_i}{2} + \frac{a_{i+1} + d_{i+1}}{2} \right) \bar{y}_i \right. \\ &\quad \left. + \frac{a_{i+1} + d_{i+1}}{2} \bar{y}_{i+1} - \frac{\Delta d_i}{\gamma} \bar{f}_{i-1} - \frac{\Delta d_{i+1}}{\gamma} \bar{f}_i \right], \quad i = 1, \dots, N-1 \\ (F\bar{y})_N &:= 0. \end{aligned}$$

Here we use the maximum norm

$$\|u\|_\infty = \max_{0 \leq i \leq N} |u_i|, \quad (10)$$

for any vector $u = (u_0, u_1, \dots, u_n)^T \in \mathbb{R}^{N+1}$ and the corresponding matrix norm.

Theorem 2.1. [10] The discrete problem (9) for $\gamma \geq f_y$, has the unique solution $\bar{y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{N-1}, \bar{y}_N)^T$, with $\bar{y}_0 = \bar{y}_N = 0$. Moreover, the following stability inequality holds

$$\|w - v\|_\infty \leq \frac{1}{m} \|Fw - Fv\|_\infty, \quad (11)$$

for any vectors $v = (v_0, v_1, \dots, v_N)^T \in \mathbb{R}^{N+1}$, $w = (w_0, w_1, \dots, w_N)^T \in \mathbb{R}^{N+1}$.

The mesh that will be used here is a modified Shishkin mesh from [13, 14], which has a greater smoothness compared to the generating function. Before the construction of the mesh, we are stating a theorem about the decomposition and estimates of the derivatives, which is necessary for the construction and further analysis.

Theorem 2.2. [20] The solution y to the problem (1)–(3) can be represented in the following way

$$y = r + s, \quad (12)$$

where for $i = 0, 1, \dots, k$ and $x \in [0, 1]$ we have that

$$|r^{(i)}(x)| \leq C, \quad (13a)$$

$$|s^{(i)}(x)| \leq C\epsilon^{-i} \left(e^{-\frac{x}{\epsilon} \sqrt{m}} + e^{-\frac{1-x}{\epsilon} \sqrt{m}} \right). \quad (13b)$$

Let $N + 1$ be the number of mesh points, $q \in (0, 1/2)$ and $\sigma > 0$ be the mesh parameter. We will define the transition point of the Shishkin mesh with

$$\lambda := \min \left\{ \frac{\sigma \epsilon}{\sqrt{m}} \ln N, q \right\}.$$

Let $\sigma = 2$.

Remark 2.3. For the sake of simplicity in representation, we assume that $\lambda = 2\epsilon(\sqrt{m})^{-1} \ln N$, as otherwise the problem can be analysed in the classical way. We shall also assume that qN is an integer. This is easily achieved by choosing $q = 1/4$ and N divisible by 4 for example.

The mesh $\Delta : x_0 < x_1 < \dots < x_N$ is generated by $x_i = \varphi(i/N)$ with the mesh generating function

$$\varphi(t) := \begin{cases} \frac{\lambda}{q}t & t \in [0, q], \\ p(t-q)^3 + \lambda & t \in [q, 1/2], \\ 1 - \varphi(1-t) & t \in [1/2, 1], \end{cases} \quad (14)$$

where p is chosen so that $\varphi(1/2) = 1/2$, i.e. $p = \frac{1}{2}(1 - \frac{\lambda}{q})(\frac{1}{2} - q)^{-3}$. Note that $\varphi \in C^1[0, 1]$ with $\|\varphi'\|_\infty, \|\varphi''\|_\infty \leq C$. Therefore the mesh sizes $h_i = x_{i+1} - x_i$, $i = 0, 1, \dots, N-1$ satisfy

$$h_i \leq \frac{C}{N} \quad \text{and} \quad |h_{i+1} - h_i| \leq \frac{C}{N^2}, \quad (15)$$

see [14] for details.

Theorem 2.4. [10] The difference scheme (8) on the mesh generated by the function (14) is uniformly convergent with respect to ε and

$$\max_{0 \leq i \leq N} |y(x_i) - \bar{y}_i| \leq C \frac{\ln^2 N}{N^2},$$

where $y(x)$ is the solution of the problem (1)–(3), \bar{y} is the corresponding numerical solution of (9), and $C > 0$ is a constant independent of N and ε .

3. Main results

On the interval $[x_i, x_{i+1}]$ using the representation (5), we look for an approximate solution in the following form

$$\tilde{y}_i(x) = \bar{y}_i u_i^I(x) + \bar{y}_{i+1} u_i^{II}(x) + \bar{\psi}_i \int_{x_i}^{x_{i+1}} G_i(x, s) \, ds, \quad i = 0, \dots, N-1, \quad (16)$$

where

$$\bar{\psi}_i = \psi((x_i + x_{i+1})/2, (\bar{y}_i + \bar{y}_{i+1})/2), \quad i = 0, 1, \dots, N-1. \quad (17)$$

We obtain that it is

$$\int_{x_i}^{x_{i+1}} G_i(x, s) \, ds = -\frac{\sinh(\beta(x_{i+1} - x))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x - x_i)) - 1] - \frac{\sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x_{i+1} - x)) - 1], \quad i = 0, \dots, N-1. \quad (18)$$

We are looking for an approximate solution on $[0, 1]$ in the form

$$Y(x) \Big|_{[x_i, x_{i+1}]} = \tilde{y}_i(x), \quad i = 0, \dots, N-1. \quad (19)$$

Using the maximum norm, we estimate the difference between the exact solution of the problem (1)–(3) and approximate solutions given by (19). This difference will be estimated on each interval $[x_i, x_{i+1}]$, $i = 0, \dots, N-1$. Taking into account (5), (16) and (19), we have that

$$|y_i(x) - \tilde{y}_i(x)| \leq |y_i - \bar{y}_i| |u_i^I(x)| + |y_{i+1} - \bar{y}_{i+1}| |u_i^{II}(x)| + \left| \int_{x_i}^{x_{i+1}} G_i(x, s) (\psi(s, y(s)) - \bar{\psi}_i) \, ds \right|, \quad i = 0, \dots, N-1. \quad (20)$$

Remark 3.1. An estimate of the value of difference $|y(x) - Y(x)|$, $\forall x \in [0, 1]$, or estimate of the error will be done for $[0, 1/2]$. An analogue estimate would hold on $[1/2, 1]$.

Note that $e^{-x\sqrt{m}/\varepsilon} \geq e^{-(1-x)\sqrt{m}/\varepsilon}$ and $h_{i+1} \geq h_i$ for $x \in [0, 1/2]$ and $e^{-x\sqrt{m}/\varepsilon} \leq e^{-(1-x)\sqrt{m}/\varepsilon}$ and $h_{i+1} \leq h_i$ for $x \in [1/2, 1]$.

Let us first estimate $\int_{x_i}^{x_{i+1}} G_i(x, s) ds$ for $x \in [0, \lambda]$.

Lemma 3.2. For $x \in [x_i, x_{i+1}]$, $i = 0, \dots, N/4 - 1$, we have the following estimate

$$\left| \frac{\sinh(\beta(x_{i+1} - x))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x - x_i)) - 1] + \frac{\sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x_{i+1} - x)) - 1] \right| \leq \frac{C \ln^2 N}{N^2}. \quad (21)$$

Proof.

$$\begin{aligned} & \frac{\sinh(\beta(x_{i+1} - x))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x - x_i)) - 1] + \frac{\sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x_{i+1} - x)) - 1] \\ &= \frac{\sinh(\beta(x_{i+1} - x_i)) - \sinh(\beta(x_{i+1} - x)) - \sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} \\ &= \frac{\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) - \beta(x_{i+1} - x) - \frac{\beta^3(x_{i+1} - x)^3}{6} - \mathcal{O}_2(\beta^5(x_{i+1} - x)^5)}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]} \\ & \quad - \frac{\beta(x - x_i) + \frac{\beta^3(x - x_i)^3}{6} + \mathcal{O}_3(\beta^5(x - x_i)^5)}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]} \\ &= \frac{\frac{1}{2} \beta^3 (x - x_i)(x - x_{i+1})(x_i - x_{i+1})}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]} \\ & \quad + \frac{\mathcal{O}_1(\beta^5 h_i^5) - \mathcal{O}_2(\beta^5(x_{i+1} - x)^5) - \mathcal{O}_3(\beta^5(x - x_i)^5)}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]}. \end{aligned}$$

Furthermore, based on the value of parameter β and the properties of the mesh, we have that

$$\begin{aligned} & \left| \frac{\frac{1}{2} \beta^3 (x - x_i)(x - x_{i+1})(x_i - x_{i+1})}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]} \right. \\ & \quad \left. + \frac{\mathcal{O}_1(\beta^5 h_i^5) - \mathcal{O}_2(\beta^5(x_{i+1} - x)^5) - \mathcal{O}_3(\beta^5(x - x_i)^5)}{\gamma \left[\beta h_i + \frac{\beta^3 h_i^3}{6} + \mathcal{O}_1(\beta^5 h_i^5) \right]} \right| \\ & \leq C_1 \frac{\frac{\ln^3 N}{N^3} + \frac{\ln^5 N}{N^5}}{\frac{\ln N}{N}} \leq \frac{C \ln^2 N}{N^2}. \quad (22) \end{aligned}$$

Now, using (22), we obtain (21). \square

Lemma 3.3. For $x \in [x_i, x_{i+1}]$, $i = N/4, \dots, N/2 - 1$, we have the following estimate

$$\left| \frac{\sinh(\beta(x_{i+1} - x))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x - x_i)) - 1] + \frac{\sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x_{i+1} - x)) - 1] \right| \leq C. \quad (23)$$

Proof. In the proof of the Lemma 3.2, it is shown that

$$\begin{aligned} & \frac{\sinh(\beta(x_{i+1} - x))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x - x_i)) - 1] \\ & + \frac{\sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} [\cosh(\beta(x_{i+1} - x)) - 1] \\ & = \frac{\sinh(\beta(x_{i+1} - x_i)) - \sinh(\beta(x_{i+1} - x)) - \sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)}. \end{aligned} \quad (24)$$

We get that

$$\begin{aligned} & \left| \frac{\sinh(\beta(x_{i+1} - x_i)) - \sinh(\beta(x_{i+1} - x)) - \sinh(\beta(x - x_i))}{\gamma \sinh(\beta h_i)} \right| \\ & \leq \frac{1}{\gamma} \left(1 + \left| \frac{\sinh(\beta(x_{i+1} - x))}{\sinh(\beta h_i)} \right| + \left| \frac{\sinh(\beta(x - x_i))}{\sinh(\beta h_i)} \right| \right) \leq C. \end{aligned} \quad (25)$$

□

Theorem 3.4. Let y be the exact solution of the problem (1)–(3), and Y be the appropriate approximate solution given in (19). We have the following estimate

$$\max_x |y(x) - Y(x)| \leq C \begin{cases} \frac{\ln^2 N}{N^2}, & x \in [0, \lambda], \\ \frac{1}{N}, & x \in [\lambda, 1 - \lambda], \\ \frac{\ln^2 N}{N^2}, & x \in [1 - \lambda, 1], \end{cases} \quad (26)$$

where the constant C does not depend on the perturbation parameter ε nor N .

Proof. We divide $[0, 1]$ by the mesh points x_i , $i = 1, \dots, N - 1$ into subintervals $[x_i, x_{i+1}]$, $i = 0, \dots, N$. Since $Y(x) = \tilde{y}_i(x)$ on $[x_i, x_{i+1}]$, we estimate the difference $|y(x) - \tilde{y}_i(x)|$ on each subinterval $[x_i, x_{i+1}]$. Based on representations of the exact solution (5) and the approximate solution (16) on the interval $[x_i, x_{i+1}]$, we have that the estimate (20) holds and

$$|y_i(x) - \tilde{y}_i(x)| \leq |y_i - \bar{y}_i| u_i^I(x) + |y_{i+1} - \bar{y}_{i+1}| u_i^{II}(x) + \left| \int_{x_i}^{x_{i+1}} G_i(x, s) [\psi(s, y(s)) - \bar{\psi}_i] ds \right|. \quad (27)$$

Let us first estimate the difference $\psi(x, y(x)) - \bar{\psi}_i$ on the interval $[x_i, x_{i+1}]$, $i = 0, \dots, N/4 - 1$, which appears in the integrand in (27). Using Lagrange's theorem we obtain

$$\begin{aligned} |\psi(x, y(x)) - \bar{\psi}_i| & = \left| f(x, y(x)) - f\left(\frac{x_i + x_{i+1}}{2}, \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) - \gamma \left(y(x) - \frac{\bar{y}_i + \bar{y}_{i+1}}{2} \right) \right| \\ & = \left| \left(\frac{\partial f(\xi, \eta)}{\partial y} - \gamma \right) \left(y(x) - \frac{\bar{y}_i + \bar{y}_{i+1}}{2} \right) + \frac{\partial f(\xi, \eta)}{\partial x} \left(x - \frac{x_i + x_{i+1}}{2} \right) \right| \\ & \leq \frac{C \ln N}{N}. \end{aligned} \quad (28)$$

Let now $i = N/4 + 1, \dots, N/2 - 1$. We have that

$$\begin{aligned} |\psi(x, y(x)) - \bar{\psi}_i| &= \left| f(x, y(x)) - f\left(\frac{x_i + x_{i+1}}{2}, \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) - \gamma \left(y(x) - \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) \right| \\ &= \left| \left(\frac{\partial f(\xi, \eta)}{\partial y} - \gamma \right) \left(y(x) - \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) + \frac{\partial f(\xi, \eta)}{\partial x} \left(x - \frac{x_i + x_{i+1}}{2}\right) \right| \\ &\leq \frac{C}{N}, \end{aligned} \quad (29)$$

where $\xi \in (x, (x_i + x_{i+1})/2)$ or $\xi \in ((x_i + x_{i+1})/2, x)$ in (28), and $\eta \in (y, (y_i + y_{i+1})/2)$ or $\eta \in ((y_i + y_{i+1})/2, y)$ in (29).

Let us estimate another difference $\psi(x, y(x)) - \bar{\psi}_i$ on the interval $[N/4, N/4 + 1]$. Since $\varepsilon^2 y''(x) = f(x, y(x))$, we get the estimate

$$\left| f(x, y(x)) - f\left(\frac{x_i + x_{i+1}}{2}, \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) \right| \leq |f(x, y(x))| + \left| f\left(\frac{x_i + x_{i+1}}{2}, \frac{\bar{y}_i + \bar{y}_{i+1}}{2}\right) \right| \leq \frac{C}{N^2}. \quad (30)$$

Now, from $|y(x_i) - \bar{y}_i| \leq \frac{C \ln^2 N}{N^2}$, $i = 0, \dots, N$, and decomposition and estimates from Theorem 2.2, we get the following estimate

$$\begin{aligned} \left| y(x) - \frac{\bar{y}_i + \bar{y}_{i+1}}{2} \right| &\leq \left| y(x) - \frac{y(x_i) + y(x_{i+1})}{2} \right| + \frac{C_1 \ln^2 N}{N^2} \\ &\leq \left| s(x) - \frac{s(x_i) + s(x_{i+1})}{2} \right| + \left| r(x) - \frac{r(x_i) + r(x_{i+1})}{2} \right| + \frac{C_1 \ln^2 N}{N^2} \\ &\leq \frac{C_2}{N^2} + |r'(\mu)| \left(x - \frac{x_i + x_{i+1}}{2}\right) + \frac{C_1 \ln^2 N}{N^2} \leq \frac{C}{N}, \end{aligned} \quad (31)$$

where $\mu \in (x, (x_i + x_{i+1})/2)$ or $\mu \in ((x_i + x_{i+1})/2, x)$. Now from (18), Lemma 3.2, Lemma 3.3, and the estimates (28), (29), (30) and (31) the assertion of the theorem follows. \square

According to the proof of the previous theorem it is shown that the difference between the exact and approximate solution $|y(x) - Y(x)|$ on $[0, \lambda]$ is of the order $O(\ln^2 N/N^2)$, while on $[\lambda, 1 - \lambda]$ that order of the error is $O(1/N)$. Based on the Theorem 2.4, the difference between the exact and the approximate solution on the mesh points is of order $O(\ln^2 N/N^2)$. In order to get the approximate solution with a satisfactory value of the error, we must conduct the correction of the approximate solutions given in (16). Namely, since this constructed approximate solution performs well at the layer, which is the most problematic part of the analysis, we will take on this part the approximate solution which was given in (16). In the remaining part of the observed domain, i.e. for $x \in [\lambda, 1 - \lambda]$ we will use a piecewise linear function.

Therefore, for $x \in [0, \lambda] \cup [1 - \lambda, 1]$, we use

$$\tilde{y}_i(x) = \bar{y}_i u_i^I(x) + \bar{y}_{i+1} u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x, s) \psi(x_i, \bar{y}) \, ds, \quad (32)$$

while for $x \in [\lambda, 1 - \lambda]$, we use the following interpolation polynomial

$$\bar{p}(x) = \begin{cases} \bar{p}_{N/4}(x) & x \in [x_{N/4}, x_{N/4+1}], \\ \vdots & \\ \bar{p}_i(x) & x \in [x_i, x_{i+1}], \\ \vdots & \\ \bar{p}_{3N/4-1}(x) & x \in [x_{3N/4-1}, x_{3N/4}], \end{cases} \quad (33)$$

where

$$\bar{p}_i(x) = \begin{cases} \frac{\bar{y}_{i+1} - \bar{y}_i}{x_{i+1} - x_i}(x - x_i) + \bar{y}_i & x \in [x_i, x_{i+1}], \\ 0 & x \notin [x_i, x_{i+1}] \end{cases} \quad (34)$$

and \bar{y}_i , $i = N/4, \dots, 3N/4 - 1$ are the already calculated values of the approximate solutions in the mesh points. Now, the approximate solution to the problem (1)–(3), has the following form

$$\tilde{Y}(x) = \begin{cases} \tilde{y}_i(x) & x \in [0, \lambda], \\ \bar{p}(x) & x \in [\lambda, 1 - \lambda], \\ \tilde{y}_i(x) & x \in [1 - \lambda, 1]. \end{cases} \quad (35)$$

Remark 3.5. In the following theorem, the estimate of the error will be calculated only for $x \in [\lambda, 1/2]$, i.e. for the value of the indexes $i = N/4, \dots, N/2$. We use the same assumptions as previously listed in Remark 3.1.

Theorem 3.6. The following estimate of the error between the exact and approximate solution (1)–(3) holds:

$$\max_{x \in [0,1]} |y(x) - \tilde{Y}(x)| \leq \frac{C \ln^2 N}{N^2}. \quad (36)$$

Proof. The case of $x \in [0, \lambda]$ has already been proved in the Theorem 3.4.

Let us show now (36) on $[\lambda, 1/2]$. Let us denote by p a polynomial which is defined in the same way as the polynomial \bar{p} in (33)–(34). The polynomial p will pass through the points with coordinates (x_i, y_i) and (x_{i+1}, y_{i+1}) , (y_i and y_{i+1} are values of the exact solution in the mesh points, i.e. $y_i = y(x_i)$, $y_{i+1} = y(x_{i+1})$). We have that

$$|y(x) - \bar{p}(x)| = |y(x) - p(x) + p(x) - \bar{p}(x)| \leq |y(x) - p(x)| + |p(x) - \bar{p}(x)|. \quad (37)$$

On every interval $[x_i, x_{i+1}]$, $i = N/4, \dots, N/2$, we get that

$$\begin{aligned} p(x) - \bar{p}(x) &= \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i) + y_i - \frac{\bar{y}_{i+1} - \bar{y}_i}{x_{i+1} - x_i}(x - x_i) - \bar{y}_i \\ &= \frac{y_{i+1} - \bar{y}_{i+1} - (y_i - \bar{y}_i)}{x_{i+1} - x_i}(x - x_i) - (y_i - \bar{y}_i), \end{aligned} \quad (38)$$

therefore in view of the Theorem 2.4 we obtain the estimate

$$|p(x) - \bar{p}(x)| \leq \frac{C \ln^2 N}{N^2}, \quad i = N/4, \dots, N/2. \quad (39)$$

In the part of the mesh when $i = N/4 + 1, \dots, N/2$, on basis of [12, Example 8.12], (13a), (13b) and (15), we obtain

$$|y - p_i(x)| \leq \frac{h^2}{8} \max_{\eta \in [x_i, x_{i+1}]} |y''(\eta)| \leq \frac{C}{N^2}. \quad (40)$$

For $i = N/4$, according to the decomposition (12) from Theorem 2.2, we obtain

$$y - p_i(x) = y - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i) + y_i \quad (41)$$

$$= s - \frac{s_{i+1} - s_i}{x_{i+1} - x_i}(x - x_i) + s_i + r - \frac{r_{i+1} - r_i}{x_{i+1} - x_i}(x - x_i) + r_i. \quad (42)$$

For the layer component, on the basis of the estimate (13b) we obtain

$$\left| s - \frac{s_{i+1} - s_i}{x_{i+1} - x_i} (x - x_i) + s_i \right| \leq |s| + |s_{i+1} - s_i| + |s_i| \\ \leq C_1 \left[e^{-\frac{x}{\varepsilon} \sqrt{m}} + e^{-\frac{1-x}{\varepsilon} \sqrt{m}} + \left(e^{-\frac{x_{i+1}}{\varepsilon} \sqrt{m}} + e^{-\frac{1-x_{i+1}}{\varepsilon} \sqrt{m}} \right) \right. \\ \left. + 2 \left(e^{-\frac{x_i}{\varepsilon} \sqrt{m}} + e^{-\frac{1-x_i}{\varepsilon} \sqrt{m}} \right) \right] \leq \frac{C}{N^2}. \quad (43)$$

For the regular component we apply again the estimate from [12, Example 8.12], and on the basis of (13a) we get that

$$\left| r - \frac{r_{i+1} - r_i}{x_{i+1} - x_i} (x - x_i) + r_i \right| \leq \frac{h^2}{8} \max_{\eta \in [x_i, x_{i+1}]} |y''(\eta)| \leq \frac{C}{N^2}. \quad (44)$$

Now, from (39), (40), (43) and (44), and the part of the proof of Theorem 3.4, which is related to $x \in [0, \lambda]$, we obtain (36). \square

4. Numerical Experiments

In this section the theoretical results of the previous section will be checked on the following example

$$\varepsilon^2 y'' = y + (1 - 2x)^2 - 8\varepsilon^2, \quad x \in (0, 1), \quad y(0) = 0, \quad y(1) = 0. \quad (45)$$

The exact solution of the test example (45) is

$$y(x) = \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}} + 4x(1 - x) - 1. \quad (46)$$

First we will calculate a discrete approximate solution, i.e. the value of approximate solutions in the mesh points, using the difference scheme (8) and then based on those results we will construct approximate solutions (16) and (35). Plots of exact and approximate solutions (16) and (35) are represented by Figure 1 and Figure 2, while the values of errors are presented in na Figure 3.

The system of equations is solved by Newton's method with initial guess $y_0 = -0.5$. The value of the constant $\gamma = 1$ has been chosen so that the condition $\gamma \geq f_y(x, y)$, $\forall (x, y) \in [0, 1] \times \mathbb{R}$ is satisfied. Because of the fact that we know the exact solution, we define the computed error E_N and the computed rate of convergence Ord in the usual way

$$E_N = \max_{0 \leq i \leq N} |y(x_i) - \bar{y}^N(x_i)|, \quad \text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln \frac{2k}{k+1}},$$

where $N = 2^k$, $k = 5, 6, \dots, 11$, $\bar{y}^N(x_i)$ is the numerical solution on a mesh with N subintervals. Values E_N and Ord are represented in the following table.

The explanations about the figures. In Figure (1a), (1c) and (1e), the plots of the exact solution of the problem (1)–(3) and the approximate solutions (16) are presented, for the values of the parameters $N = 32$ and $\varepsilon = 2^{-4}, 2^{-6}, 2^{-10}$, respectively, while in figures (1b), (1d) and (1f) graphics of exact and numerical solution (16) were given for the values of the parameters $N = 64, 128, 256$ and $\varepsilon = 2^{-10}$, respectively. In figure (1a), (1c) and (1e) one can notice an increase of the error value, or differences in the graphs between the exact and numerical solutions, while in Figure (1a) it is very difficult to distinguish between the exact and numerical solutions (16), in Figure (1e) the deviation between the numerical and exact solution can be seen. From the presented graphs it is evident that there is a decrease of the error value due to an increase in the number of points N .

N	E_n	Ord	E_n	Ord	E_n	Ord
2^5	$4.9836e-03$	2.01	$1.8622e-02$	2.95	$1.9923e-02$	2.55
2^6	$1.7834e-03$	1.98	$4.1194e-03$	2.01	$5.4155e-03$	2.00
2^7	$6.1200e-04$	2.00	$1.3925e-03$	2.00	$1.8429e-03$	2.00
2^8	$1.9982e-04$	2.00	$4.5548e-04$	2.00	$6.0172e-04$	2.00
2^9	$6.3269e-05$	2.00	$1.4417e-04$	2.00	$1.9039e-04$	2.00
2^{10}	$1.9527e-05$	2.00	$4.4492e-05$	2.00	$5.8762e-05$	2.00
2^{11}	$5.9069e-06$	–	$1.3460e-05$	–	$1.7776e-05$	–
ε	2^{-4}		2^{-6}		2^{-10}	

N	E_n	Ord	E_n	Ord	E_n	Ord
2^5	$1.9969e-02$	2.43	$1.9957e-02$	2.41	$1.9957e-02$	2.41
2^6	$5.7712e-03$	2.02	$5.8271e-03$	2.02	$5.8271e-03$	2.02
2^7	$1.9427e-03$	2.00	$1.9616e-03$	2.00	$1.9616e-03$	2.00
2^8	$6.4337e-04$	2.00	$6.4051e-04$	2.00	$6.4051e-04$	2.00
2^9	$2.0072e-04$	2.00	$2.0266e-04$	2.00	$2.0266e-04$	2.00
2^{10}	$6.1950e-05$	2.00	$6.2550e-05$	2.00	$6.2550e-05$	2.00
2^{11}	$1.8740e-05$	–	$1.8921e-05$	–	$1.8921e-05$	–
ε	2^{-12}		2^{-20}		2^{-30}	

Table 1: Errors E_N and convergence rates Ord for approximate solutions.

In Figures (2a), (2c) and (2e) the plots of the exact (1)–(3) and approximate solution (35) are given. For the calculation of the approximate solutions we used $N = 32$ points, while the value of the perturbation parameter was $\varepsilon = 2^{-4}, 2^{-6}, 2^{-10}$, respectively. From the presented graphics it can be seen a decrease of perturbation parameter ε , with a constant value of the number of points N a value of the error is slightly increasing. However, this increase is smaller than in the case of use of approximate solutions (16). In the Figure (2b), (2d) and (2f) there are graphs of the correct solution of the problems (1)–(3) and approximate solutions. Graphs on all three figures are obtained for a fixed value of parameter ε , while approximate solution is obtained by using $N = 64, 128, 256$ number of points, respectively.

In Figures (3a), (3c) and (3e) the plots of the error of the approximate solutions (16) are represented, while in Figures (3b), (3d) and (3f) are graphs of the error of the approximate solution (35). Side by side are graphs of the errors of the approximate solution, to the left is (16), while on the right are approximate solution (35) for the same values of the parameter ε and N . From the graph we can see that values of the error agree with the theoretical results. In the graph, on the right side is a value of the error from the order $\mathcal{O}(N^{-1})$, while on the graphs from the right side is a value of the error from the order $\mathcal{O}(N^{-2} \ln^2 N)$, and therefore in this way we have a confirmation of the theoretical results.

5. Conclusion

In this paper we performed a construction of approximate solutions for singularly-perturbed boundary value problem (1)–(3). First, we calculated a discrete approximate solution, i.e. the value of approximate solution in points of the mesh, and then we constructed an approximate solution by using a representation of the exact solution via Green's functions. Order of the value of the error is $\mathcal{O}(N^{-1})$ in the maximum norm. The basis functions are exponential. From Theorem 3.4 we can see that the value of errors in this way constructed approximate solution is in the part of the domain where lies boundary layer of order $\mathcal{O}(\ln^2 N/N^2)$, while out of the layer are of order $\mathcal{O}(1/N)$. In order to gain the approximate solution with the smallest error, basis function of the exponential type of the outer boundary layer is replaced with linear functions. Error in this case is in the order $\mathcal{O}(\ln^2 N/N^2)$, also in the maximum norm.

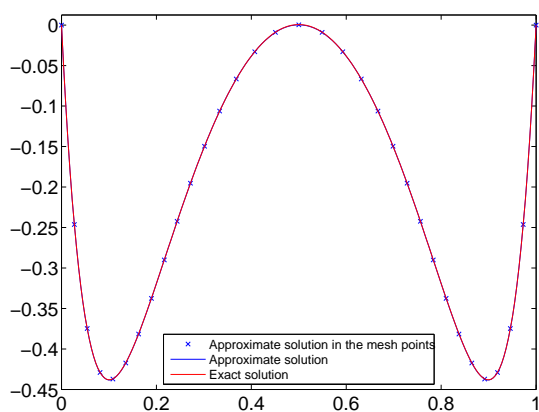
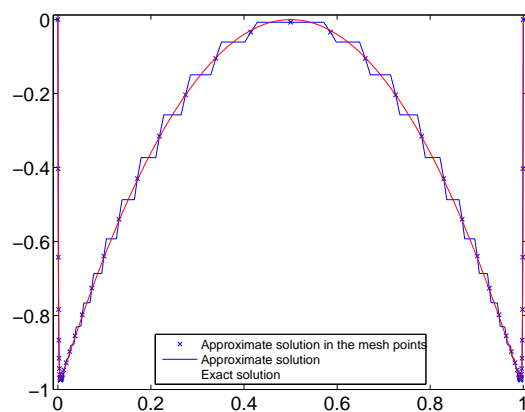
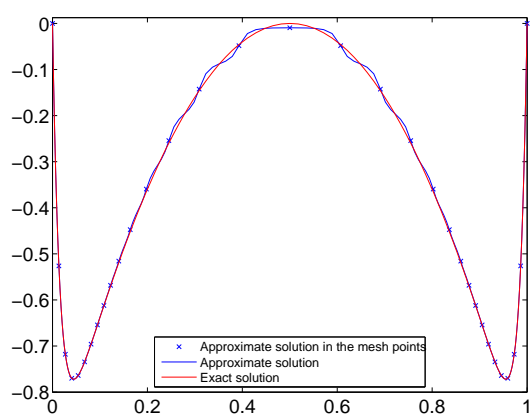
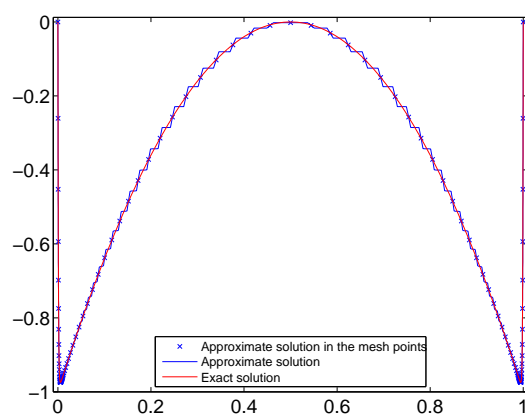
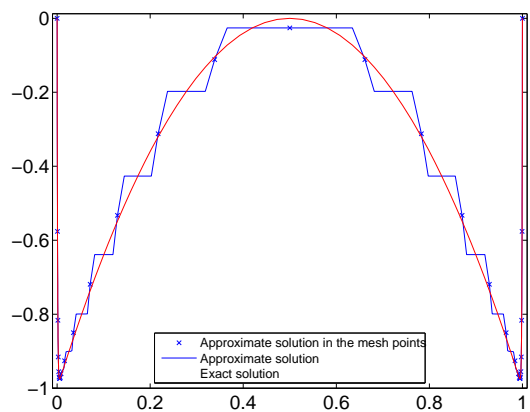
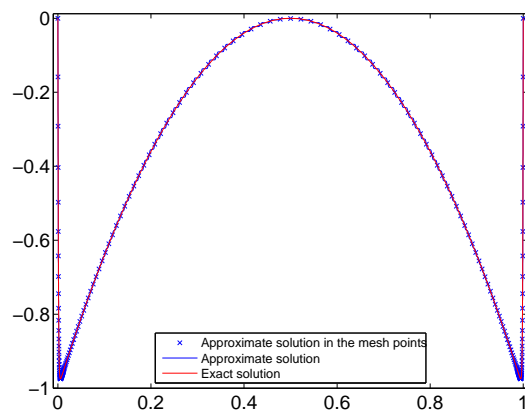
(a) $N = 32, \varepsilon = 2^{-4}$ (b) $N = 64, \varepsilon = 2^{-10}$ (c) $N = 32, \varepsilon = 2^{-6}$ (d) $N = 128, \varepsilon = 2^{-10}$ (e) $N = 32, \varepsilon = 2^{-10}$ (f) $N = 256, \varepsilon = 2^{-10}$

Figure 1

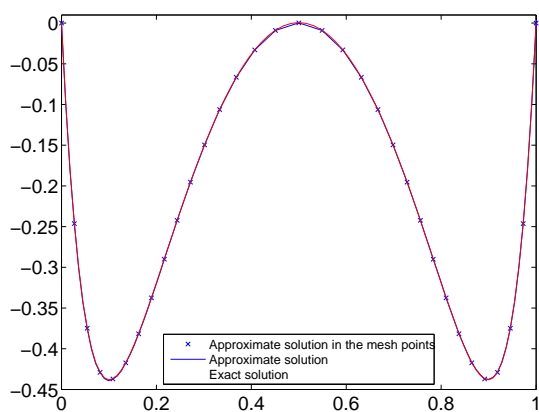
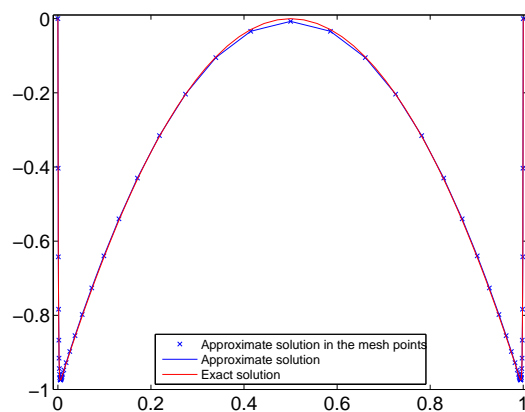
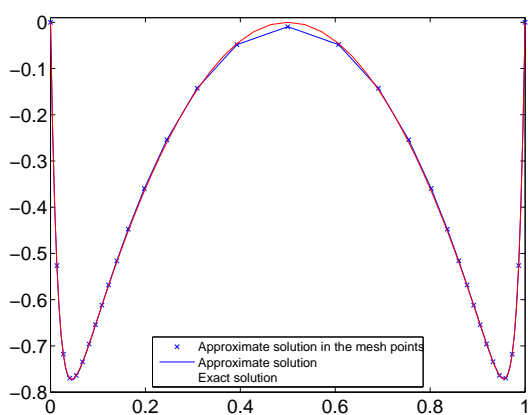
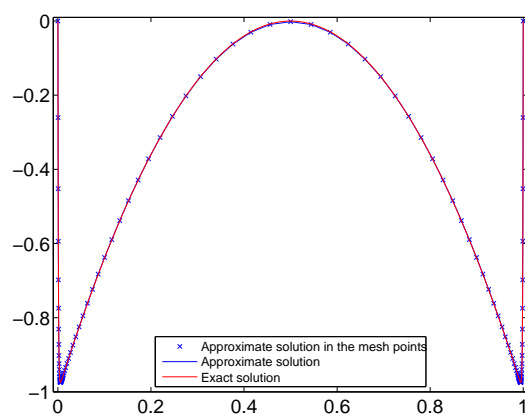
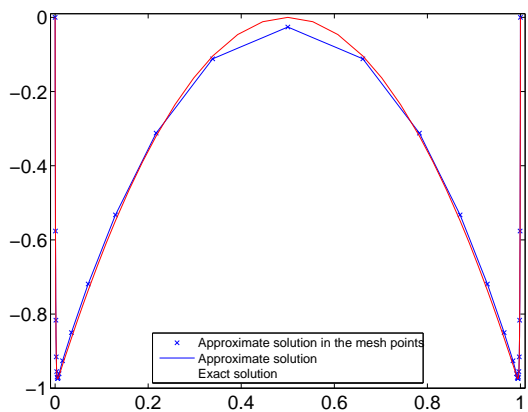
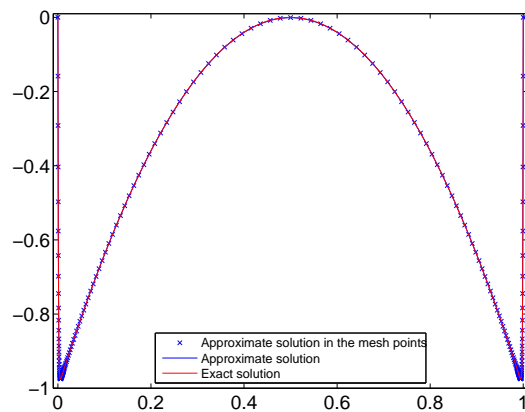
(a) $N = 32, \varepsilon = 2^{-4}$ (b) $N = 64, \varepsilon = 2^{-4}$ (c) $N = 32, \varepsilon = 2^{-6}$ (d) $N = 128, \varepsilon = 2^{-10}$ (e) $N = 32, \varepsilon = 2^{-10}$ (f) $N = 256, \varepsilon = 2^{-10}$

Figure 2

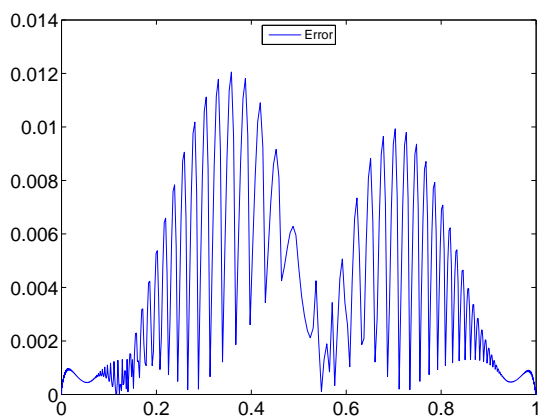
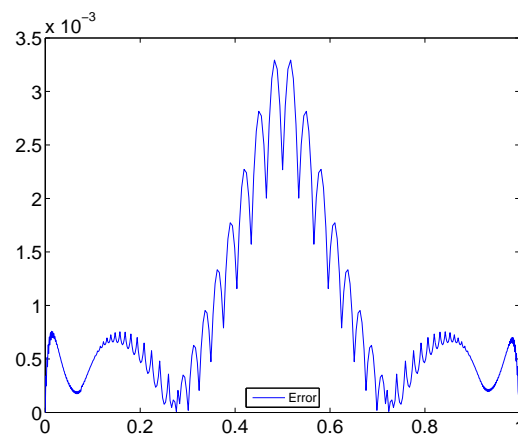
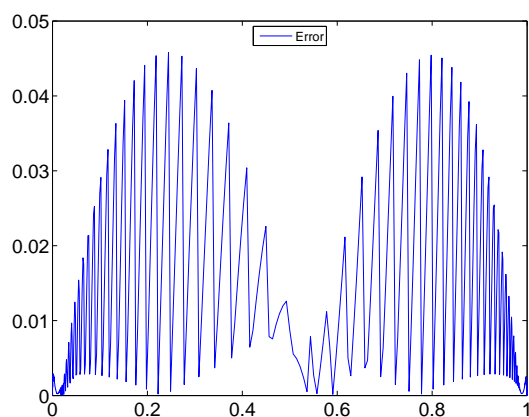
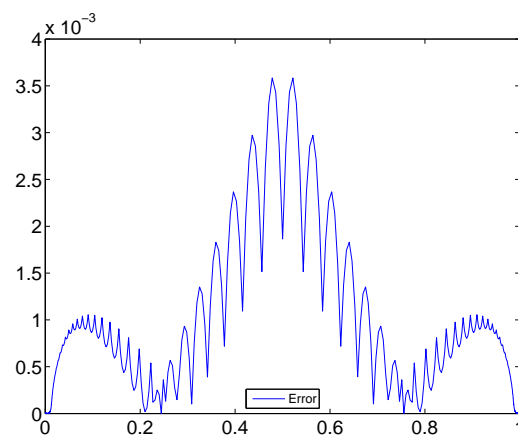
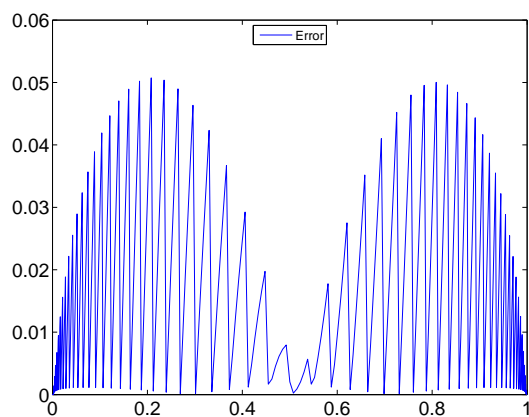
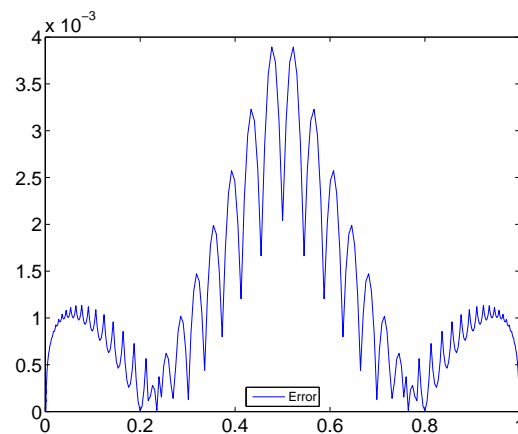
(a) $N = 128, \varepsilon = 2^{-6}$ (b) $N = 128, \varepsilon = 2^{-6}$ (c) $N = 128, \varepsilon = 2^{-9}$ (d) $N = 128, \varepsilon = 2^{-9}$ (e) $N = 128, \varepsilon = 2^{-12}$ (f) $N = 128, \varepsilon = 2^{-12}$

Figure 3: The graphs error

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