Torsion Gravitational Waves and Spectral Analysis of the Massless Dirac Operator†

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† Joint work with Elvis Baraković

Structure of talk

- Introduction and motivation
- Extended theories of gravity
- pp-waves with torsion
- Physical interpretation
- Spectral (a)symmetry of the massless Dirac operator

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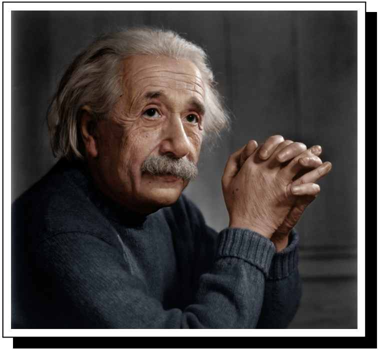
Albert Einstein (1879 - 1955)

The ordinary adult never gives a thought to spacetime problems ...

I, on the contrary, developed so slowly that I did not begin to wonder about space and time until I was an adult.

I then delved more deeply into the problem than any other adult or child would have done.



Newtonian gravity

Mass if the source of the gravitational *field*.

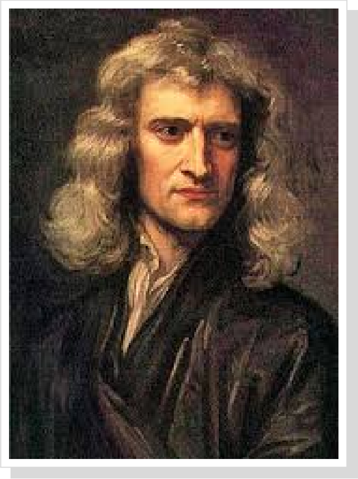
Analytical form

$$\vec{F} = -G \, \frac{m_1 m_2}{|\vec{r_2} - \vec{r_1}|^3} \, (\vec{r_2} - \vec{r_1}),$$

$$G = 6.67 \times 10^{-11} Nm^2/kg^2$$

Newton's gravitational constant.

Gravity a *force*.



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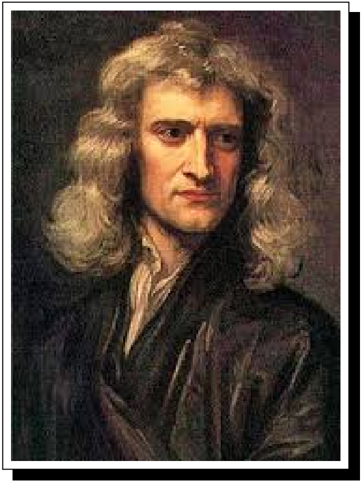

General relativity (GR)

Einstein unhappy - laws of physics should be invariant wrt choice of inertial frame. Fundamental laws of physics maintain the same form under coordinate transformations.

Spacetime considered a connected real 4–manifold, equipped with a (Lorentzian) metric g (pseudo-Riemannian).

Einstein-Hilbert action

$$\frac{c^4}{16\pi G}\int \mathcal{R}.$$

Einstein's equations

$$\underbrace{Ric_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}}_{GEOMETRY} = \underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{MATTER}.$$

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Consequences and tests of Einstein's GR

- Precession of the perihelion of Mercury;
- Deflection of light near the sun;
- Gravitational redshift;
- Temporal delay of radar;
- Global Positioning System (GPS);
- Black holes;
- Expansion of universe;
- Singularities;
- **gravitational waves.**

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Mechanical waves

- Rock thrown in water - transverse waves;
- Vibrations of vocal chords - longitudinal waves;
- Tsunami inducing earthquakes - surface waves.

Need material environment to propagate.

Particle position $u(\vec{r}, t)$ at given time t satisfies homogenous *wave equation*

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{1}{v^2}\frac{\partial}{\partial t^2}\right) u(\vec{r}, t) = 0.$$

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Electromagnetic waves

Equations of an electromagnetic field.

James Clark Maxwell 1861-62

Gauss' Law of electricity: $$\oint_{\substack{\text{closed}\\ \text{surface}}} \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

Gauss' Law of magnetism: $$\oint_{\substack{\text{closed}\\ \text{surface}}} \vec{B} \cdot d\vec{A} = 0$$

Faraday's Law: $$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$$

Ampere-Maxwell Law: $$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\phi_E}{dt} + \mu_0 i_{enc}$$

EM waves

- In space without charge or current, electric field $\vec{E}$ and magnetic field $\vec{B}$ satisfy homogenous wave equation.
- In space in which the distribution of charge and current is given, there exists an *electromagnetic* (EM) field.
- Energy of EM field transferred along a direction orthogonal to electric and magnetic field vectors - EM wave transverse.
- EM waves propagate through *vacuum* at speed of light c.
- Hertz's experiment (1888) confirmed existence.

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Gravitational waves in GR

Mathematical proof of existence of gravitational waves in GR very simple.

Take the following metric, far from the field source:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$$

where h is a small perturbation (wave) and $g^{(0)}$ is the metric of specetime where we observe the wave.

Einstein's equations become:

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{1}{c}\frac{\partial}{\partial t^2}\right) h_{\mu\nu} = 0.$$

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What is oscillating?

Mechanical wave - transfer of oscillations through material medium.

EM wave - transfer of electric and magnetic field of an accelerated charged particle.

Gravitational waves - the *metric* itself is oscillating! Distance between two infitesimal points in spacetime with metric g is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu .$$

If we take square root of RHS and integrate between points, we get arc length - distance between points in curved spacetime.

Metric oscillation means oscillation of distance between points.

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Detecting gravitational waves

1. Webber rods
   - ALLEGRO (Louisiana)
   - MiniGRAIL (Netherlands)
   - AURIGA (Italy)
2. Interferometres
   - LIGO (Lousiana, Washigton)
   - eLISA (orbit)
   - VIRGO (Italy)
3. High-frequency detectors.
   - Birmigham, Genoa, Chongking.

Detecting gravitational waves

- Detecting oscillations of the distance between bodies.
- Amplitude of gravitational wave falls with distance.
- September 14, 2015 at 5:51 EST, dual LIGO observatory of gravitational waves in Louisiana and Washington detects GW.
- Signals based on last moments of two black holes (29 and 36 times mass of Sun) merging into a massive rotating black hole, 1.3 billion years ago.
- 3 times mass of Sun converted into energy and highest energy output was 50 times greater than whole observable universe.

$$E = 5.3 \times 10^{47} J$$

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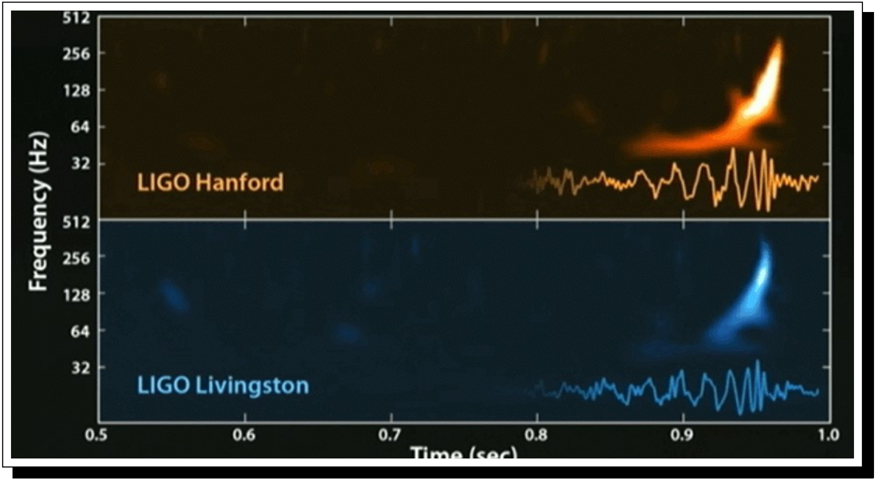
Frequency (Hz)
512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)

Extending GR

Developments in physics evoke possibility that treatment of spacetime might involve more than the Lorentzian spacetime of GR:

- Failure to quantize gravity;
- Generalisation of the 3D theory of elastic continua with microstructure to 4D spacetime of gravity;
- Description of hadron (or nuclear) matter in terms of extended structures;
- Study of early universe;
- Accelerating universe...

Smallest departure from Lorentzian spacetime of GR:

- admit torsion - Riemann–Cartan spacetime;
- admit possible nonmetricity - *metric–affine* spacetime.

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Metric–affine gravity (MAG)

Spacetime a connected real 4–manifold $\mathcal{M}$ equipped with Lorentzian metric g and affine connection Γ, i.e.

$$\nabla_\mu u^\lambda = \partial_\mu u^\lambda + \Gamma^\lambda{}_{\mu\nu} u^\nu .$$

Independent linear connection Γ distinguishes MAG from GR from start.

Metric g and connection Γ viewed completely *independently*.

Unknowns of MAG : 10 independent metric components $g_{\mu\nu}$ and 64 connection $\Gamma^\lambda{}_{\mu\nu}$ coefficients.

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Quadratic metric-affine gravity (QMAG)

Action

$$S := \int q(R)$$

$q(R)$ is $O(1,3)$ invariant quadratic form on curvature R
16 R^2 terms and 16 coupling constants.

Action conformally invariant, unlike Einstein-Hilbert.

Why quadratic forms?

Yang-Mills action

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Field equations of QMAG

We vary action wrt g and Γ *independently*.

Euler-Lagrange system of equations

$$\frac{\partial S}{\partial g} = 0, \tag{1}$$

$$\frac{\partial S}{\partial \Gamma} = 0. \tag{2}$$

10+64 nonlinear PDEs with 10+64 unknowns.

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Classical pp-waves

"Plane-fronted gravitational waves with parallel rays".

Definition

A *pp-wave* is a Riemannian spacetime which admits a nonvanishing parallel spinor field.

Definition

A *pp-wave* is a Riemannian spacetime with metric in form

$$ds^2 = 2\,dx^0\,dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3)\,(dx^3)^2$$

in some local coordinates (x^0, x^1, x^2, x^3).

Exact formula for curvature: trace-free Ricci+Weyl.

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Generalising pp-waves

Classical pp-waves with parallel Ricci: Riemannian solution of QMAG.

Generalisation: connection not necessarily Levi-Civita.

We still use pp-metric but introduce explicitly given torsion.

- Pasic and Vassiliev:
  generalised pp-waves with purely *TENSOR* torsion;
- Barakovic and Pasic:
  generalised pp-waves with purely *AXIAL* torsion.

New solutions of QMAG?

Generalising pp-waves

Classical pp-waves with parallel Ricci: Riemannian solution of QMAG.

Generalisation: connection not necessarily Levi-Civita.

We still use pp-metric but introduce explicitly given torsion.

- Pasic and Vassiliev:
  generalised pp-waves with purely *TENSOR* torsion;
- Barakovic and Pasic:
  generalised pp-waves with purely *AXIAL* torsion.

New solutions of QMAG?

PP-waves with purely tensor torsion

We view the *polarized Maxwell equation*

$$*dA = \pm \mathrm{i} dA.$$

Solutions of this equation in form of plane waves:

$$A = h(\varphi)\, m + k(\varphi)\, l\,,$$

$$\varphi : M \mapsto \mathbb{R}, \qquad \varphi(x) := \int_M l \cdot dx\,.$$

Definition

Generalised pp-waves with purely tensor torsion are metric-compatible spacetimes with pp-metric and *torsion*

$$T := \frac{1}{2}\mathrm{Re}(A \otimes dA).$$

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Generalised pp-waves with tensor torsion of parallel Ricci curvature are solutions of (1) i (2).

Very simple explicit description.

Result published in "PP-waves with torsion and metric affine gravity", V. Pasic, D. Vassiliev, *Class. Quantum Grav.* **22** 3961-3975.

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Physical interpretation

- Curvature od spacetimes split.
- Torsion and torsion generated curvature waves travelling at speed of light.
- Underlying pp-space can be viewed as ‘gravitational imprint’ of a travelling particle.
- Mathematical model for elementary particle?

MAG vs Einstein-Weyl

We view the massless Dirac action

$$S_{\mathrm{W}} := 2i \int \Big(\xi^a \, \sigma^\mu{}_{a\dot b} \, (\nabla_\mu \bar\xi^{\dot b}) \; - \; (\nabla_\mu \xi^a) \, \sigma^\mu{}_{a\dot b} \, \bar\xi^{\dot b} \Big),$$

The massless Dirac equation takes form

$$\sigma^\mu{}_{a\dot b} \nabla_\mu \xi^a - \frac{1}{2} T^\eta{}_{\eta\mu} \sigma^\mu{}_{a\dot b} \xi^a = 0.$$

There exist pp-wave type solutions of Einstein-Weyl model:

$$S_{\mathrm{EW}} := k \int \mathcal{R} + S_{\mathrm{W}},$$

$$\partial S_{\mathrm{EW}} / \partial g = 0, \quad \partial S_{\mathrm{EW}} / \partial \xi = 0.$$

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pp-waves with purely axial torsion

Definition

A generalised pp-wave with purely axial torsion is a metric compatible spacetime with pp-metric and torsion

$$T := *A$$

where A is a real vector field defined by $A = k(\varphi)\, l$, where $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function of the phase φ.

PP-waves with purely axial torsion

Curvature

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4}(k(x^3))^2 \operatorname{Re}\left((l \wedge m) \otimes (l \wedge \overline{m}) \right) \mp \frac{1}{2} k'(x^3) \operatorname{Im}\left((l \wedge m) \otimes (l \wedge \overline{m}) \right)$$

Torsion

$$T = \mp \frac{i}{2} k \; l \wedge m \wedge \overline{m}.$$

Ricci curvature is $Ric = \frac{1}{2}(f_{11} + f_{22} - k^2) l \otimes l$.

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PP-waves with purely axial torsion

- Torsion of a generalised pp-wave is purely axial.
- Connection is metric compatible.
- Curvatures generated by the Levi-Civita connection and torsion simply add up to produce the formula for curvature.
- Weyl piece of torsion generated curvature is zero.
- Torsion also generates Ricci curvature.
- Curvature has three irreducible pieces.
- Scalar and pseudoscalar curvatures are zero.
- Ricci curvature is parallel if $f_{11} + f_{22} = (k(x^3))^2 + C$.
- Ricci curvature is zero if $f_{11} + f_{22} = (k(x^3))^2$.

The first result-published

Theorem

Generalised pp-waves with purely axial torsion of parallel $\{Ric\}$ are solutions of (1), (2) for the Yang-Mills action.

Pasic V., Barakovic E. : *Torsion wave solutions in Yang–Mielke theory of gravity.* Advances in High Energy Physics, Article ID 239076:7, (2015).

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Theorem

Generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of (1), (2) for the quadratic form with 11 R^2 terms.

Proof by "brute force"! Assumptions:

- Our spacetime is metric compatible.
- Torsion is purely axial.
- Ricci curvature is symmetric.
- Scalar curvature $\mathcal{R}$ and pseudoscalar curvature $\mathcal{R}_*$ are zero.

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The second result

Explicit representation of the field equations:

$$0 = 2d_1 W^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} + d_2 \epsilon^{\eta\nu\alpha\beta} Ric_{\kappa\nu} {Ric_*}^{\kappa}{}_{\eta} - d_3 \epsilon^{\kappa\lambda\xi\alpha} {W_{\kappa\lambda\mu}}^{\beta} {Ric_*}^{\mu}{}_{\xi},$$

$$\begin{aligned}
0 = {} & d_1 \left\{ \nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + T_{\mu\eta\lambda} {Ric_\kappa}^{\eta} + T_{\mu\kappa\eta} {Ric_\lambda}^{\eta} \right\} \\
& - d_4 \left\{ (g_{\kappa\mu} \mathcal{W}^{\xi\zeta}{}_{\lambda\eta} - g_{\lambda\mu} \mathcal{W}^{\xi\zeta}{}_{\kappa\eta}) T^{\eta}{}_{\xi\zeta} + (g_{\kappa\mu} \epsilon^{\vartheta\zeta}{}_{\eta\lambda} - g_{\lambda\mu} \epsilon^{\vartheta\zeta}{}_{\eta\kappa}) T^{\eta}{}_{\xi\zeta} {Ric_*}^{\xi}{}_{\vartheta} \right\} \\
& + c_5 \left\{ \epsilon^{\eta\xi}{}_{\kappa\mu} \nabla_\xi Ric_{*\lambda\eta} - \epsilon^{\eta\xi}{}_{\lambda\mu} \nabla_\xi Ric_{*\kappa\eta} + \tfrac{1}{2} ({\epsilon_\kappa}^{\eta\xi\zeta} Ric_{*\lambda\eta} - {\epsilon_\lambda}^{\eta\xi\zeta} Ric_{*\kappa\eta}) T_{\mu\zeta\xi} \right\} \\
& - c_3 \Big\{ 2T^{\eta}{}_{\lambda\xi} \mathcal{W}^{\xi}{}_{\mu\kappa\eta} + 2T^{\eta}{}_{\xi\kappa} \mathcal{W}^{\xi}{}_{\mu\lambda\eta} + T_{\mu\xi\eta} {\mathcal{W}_{\kappa\lambda}}^{\eta\xi} + \epsilon^{\vartheta}{}_{\mu\eta\lambda} T^{\eta}{}_{\xi\kappa} {Ric_*}^{\xi}{}_{\vartheta} \\
& \qquad - \epsilon^{\vartheta}{}_{\mu\kappa\eta} T^{\eta}{}_{\xi\lambda} {Ric_*}^{\xi}{}_{\vartheta} - \epsilon^{\vartheta\xi}{}_{\eta\kappa} T^{\eta}{}_{\xi\lambda} Ric_{*\mu\vartheta} + \epsilon^{\vartheta\xi}{}_{\eta\lambda} T^{\eta}{}_{\xi\kappa} Ric_{*\mu\vartheta} \\
& \qquad + \epsilon^{\vartheta}{}_{\mu\lambda\kappa} \nabla_\xi {Ric_*}^{\xi}{}_{\vartheta} - \epsilon^{\vartheta\xi}{}_{\lambda\kappa} \nabla_\xi Ric_{*\mu\vartheta} \Big\}.
\end{aligned}$$

Physical interpretation

Use previous result: Pasic V., Barakovic E.: *PP-waves with torsion: a metric-affine model for the massless neutrino.* Gen. Relativ. Gravit. 46:1787, 2014.

We consider the complexified curvature

$$\mathfrak{R}_A := r\,(l \wedge m) \otimes (l \wedge \overline{m}),$$

$\mathfrak{R}_A$ is completely determined by rank 1 spinor $\xi = r^{1/4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The spinor field ξ satisfies the massless Dirac equation!

Generalised pp-waves with purely axial torsion of MAG and pp-wave type solutions of E-W theory are very similar.

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Physical interpretation

Conjecture

Generalised pp-waves of parallel Ricci curvature represent a metric-affine model for the massless neutrino.

- Elvis Barakovic, Vedad Pasic: *"Physical Interpretation of PP-waves with Axial Torsion"*, 14th Marcel Grossmann Meeting on General Relativity, Rome. Accepted for publication, Proceedings, World Scientific, 2017.
- Vedad Pasic, Elvis Barakovic: *"Axial Torsion Waves in Metric-affine Gravity"*, 14th Marcel Grossmann Meeting on General Relativity, Rome. Accepted for publication, Proceedings, World Scientific, 2017.

Massless Dirac operator

Matrix 2×2 operator

$$W := -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \right).$$

Euclidean Metric, unit torus $\mathbb{T}^3$:

$$W = -i \begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix}.$$

We look for eigenfunctions of form $v(x) = ue^{im_\alpha x^\alpha}$, $m \in \mathbb{Z}^3$, $u \in \mathbb{C}^2$.

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Spectrum of massless Dirac operator

Massless Dirac describes the massless neutrino.

Eigenvalues interpreted as energy levels of the massless neutrino.

Unit torus $\mathbb{T}^3$ (Euclidean metric) and the unit sphere $\mathbb{S}^3$ - the spectrum is symmetric about zero!

No physical reason for spectrum to be symmetric (Atiyah et al.).

Manifold with nontrivial topology and flat metric or manifold with trivial topology and perturbed Euclidean metric.

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Massless Dirac operator spectrum

- Zero is eigenvalue of multiplicity 2.
- For all $m \in \mathbb{Z}^3 \setminus \{0\}$ we have eigenvalue $\|m\|$ and unique eigenfunction $ue^{im_\alpha x^\alpha}$.
- For all $m \in \mathbb{Z}^3 \setminus \{0\}$ we have eigenvalue $-\|m\|$ and unique eigenfunction $ue^{im_\alpha x^\alpha}$.
- The spectrum is symmetric!
- Spectral asymmetry : trivial topology 3–torus & euclidean metric perturbation.

$$g_{\alpha\beta}(x^1;\epsilon) = \delta_{\alpha\beta} + \epsilon h_{\alpha\beta}(x^1) + \frac{\epsilon^2}{4} k_{\alpha\beta}(x^1) + O(\epsilon^3).$$

- We obtain asymptotic formulae for eigenvalues ± 1.

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We consider the eigenvalue problem

$$W_{1/2}(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon).$$

$$\begin{aligned} W_{1/2}(\epsilon) &= W^{(0)}_{1/2} + \epsilon W^{(1)}_{1/2} + \epsilon^2 W^{(2)}_{1/2} + O(\epsilon^3), \\ v(\epsilon) &= v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + O(\epsilon^3), \\ \lambda(\epsilon) &= \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + O(\epsilon^3). \end{aligned}$$

Ritz-Galerkin: we observe spectral asymmetry on $\mathbb{T}^3$.

Principal and subprincipal symbol of the perturbed massless Dirac operator on half-densities is given by the frame and coframe

$$A_1(x,\xi;\epsilon) = \begin{pmatrix} {e_3}^\alpha & {e_1}^\alpha - i{e_2}^\alpha \\ {e_1}^\alpha + i{e_2}^\alpha & -{e_3}^\alpha \end{pmatrix} \xi_\alpha,$$

$$A_{\mathrm{sub}} = \frac{3}{4}(*T^{\mathrm{ax}}(x;\epsilon))I,$$

where

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is the explicit formula for the Hodge dual of axial piece of torsion!

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Goal

Calculate $\lambda^{(1)}$ and $\lambda^{(2)}$ for the eigenvalues ± 1.

Can we use standard perturbation theory?

We developed the perturbation theory for the massless Dirac operator.

$$\lambda^{(1)} = \langle W^{(1)}_{1/2} v^{(0)}, v^{(0)} \rangle,$$
$$\lambda^{(2)} = \langle W^{(2)}_{1/2} v^{(0)}, v^{(0)} \rangle - \langle (W^{(1)}_{1/2} - \lambda^{(1)}) Q (W^{(1)}_{1/2} - \lambda^{(1)}) v^{(0)}, v^{(0)} \rangle.$$

Main result-prepared for publication

Under the arbitrary perturbation of the Euclidean metric $g(x^1;\epsilon)$, the coefficients in the asymptotic expansion of the eigenvalues ± 1

$$\lambda_+(\epsilon) = 1 + \lambda_+^{(1)}\epsilon + \lambda_+^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0,$$
$$\lambda_-(\epsilon) = -1 + \lambda_-^{(1)}\epsilon + \lambda_-^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0,$$

are calculated explicitly!

The coefficients of the linear terms

$$\lambda_+^{(1)} = -\frac{1}{2}\widehat{h}_{11}(0), \quad \lambda_-^{(1)} = \frac{1}{2}\widehat{h}_{11}(0),$$

The coefficients of the quadratic terms

$$\lambda_+^{(2)} = \frac{3}{8}\widehat{(h^2)}_{11}(0) - \frac{1}{8}\widehat{k}_{11}(0) - \frac{\mathrm{i}}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m)$$
$$- \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1}(m+1)^2\ \widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)}$$
$$- \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{31}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right)$$
$$- \frac{\mathrm{i}}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{21}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right), \qquad (3)$$

$$\lambda_-^{(2)} = -\frac{3}{8}\widehat{(h^2)}_{11}(0) + \frac{1}{8}\widehat{k}_{11}(0) - \frac{\mathrm{i}}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m)$$
$$- \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} \frac{1}{m+1}(m-1)^2\ \widehat{h}_{11}(m+1)\overline{\widehat{h}_{11}(m+1)}$$
$$- \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{31}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right)$$
$$- \frac{\mathrm{i}}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{21}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right), \qquad (4)$$

Main result

No spectral asymmetry in linear term

For an arbitrary perturbation $\lambda_+^{(1)} + \lambda_-^{(1)} = 0$.

Spectral asymmetry in quadratic term

We seek perturbations for which $\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0$.

$$\widehat{h}_{11}(m) = \widehat{h}_{21}(m) = \widehat{h}_{31}(m) = 0,$$

$$\varepsilon_{\beta\gamma 1} \sum_{m \in \mathbb{Z} \setminus \{0\}} m \overline{\widehat{h}_{\alpha\beta}(m)} \widehat{h}_{\alpha\gamma}(m) \neq 0.$$

Examples of spectral asymmetry

Example

For the perturbation matrices

$$h_{\alpha\beta}(x^1) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}, \quad k_{\alpha\beta}(x^1) = \begin{pmatrix} \sin x^1 & \cos x^1 & 0 \\ \cos x^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get that

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3), \tag{5}$$

$$\lambda_-(\epsilon) = -1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3). \tag{6}$$

Note: $\lambda_+^{(1)} = \lambda_-^{(1)} = 0, \quad \lambda_+^{(1)} + \lambda_-^{(1)} = 0$ and $\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0$.

The numerical results match the analytical!

Examples of spectral asymmetry

Example

For the perturbation matrices

$$h_{\alpha\beta}(x^1) = \begin{pmatrix} 1 & \cos x^1 & \sin x^1 \\ \cos x^1 & \cos x^1 & \sin x^1 \\ \sin x^1 & \sin x^1 & -\cos x^1 \end{pmatrix}, \quad k_{\alpha\beta}(x^1) = \begin{pmatrix} \sin x^1 & \cos x^1 & 0 \\ \cos x^1 & -\sin x^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get that

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon + \frac{3}{4}\epsilon^2 + O(\epsilon^3), \tag{7}$$

$$\lambda_-(\epsilon) = -1 + \frac{1}{2}\epsilon - \epsilon^2 + O(\epsilon^3). \tag{8}$$

Note: $\lambda_+^{(1)} \neq 0 \wedge \lambda_-^{(1)} \neq 0, \quad \lambda_+^{(1)} + \lambda_-^{(1)} = 0$ and $\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0$.

The numerical results match the analytical!

Future work

- New solutions of QMAG when the quadratic form has 16 R^2 terms;
- Generalised pp-waves with purely trace torsion;
- New solutions of QMAG in the Yang-Mills case, 11 R^2 or 16 R^2?;
- The physical interpretation;
- Analysis of the massless Dirac operator.

Thank You and welcome to Tuzla!