Purely axial torsion waves in metric–affine gravity

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Abstract

The approach of **metric–affine** field theory is viewing spacetime as a real oriented 4-manifold equipped with a metric and an independent affine connection.

The 10 independent components of the metric tensor and the 64 connection coefficients are our unknowns. We observe spacetimes whose connection is generated by torsion which is purely axial with Minkowski metric. We show that this connection is necessarily metric-compatible and provide explicit formula for the torsion of such spacetimes.

We describe and provide the full characterisation of the properties of curvature of such spacetime, presenting explicit formulae for all possible irreducible pieces of curvature.

We also provide possible applications of these general results to concrete cases of solutions of metric-affine gravity. Special attention is given to applications to the (massless) Dirac equation and operator.

Introduction and notation

Spacetime a connected real 4–manifold $\mathcal{M}$ equipped with Lorentzian metric g and affine connection Γ, $\nabla_\mu u^\lambda = \partial_\mu u^\lambda + \Gamma^\lambda{}_{\mu\nu} u^\nu$.

Smallest departure from the Riemannian spacetime of Einstein's general relativity leads to *torsion* and possible *nonmetricity*.

Independent linear connection Γ distinguishes MAG from general relativity from the beginning. Metric g and connection Γ are viewed completely *independently*.

Unknowns of MAG : 10 independent metric components $g_{\mu\nu}$ and 64 connection coefficients. We say that our connection Γ is *metric compatible* if $\nabla g \equiv 0$. Torsion is defined to be as $T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$. We define curvature as

$$R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda},$$

Ricci curvature as $Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}$, scalar curvature as $\mathcal{R} := Ric^\lambda{}_\lambda$ and trace-free Ricci curvature as $\mathcal{R}ic := Ric - \frac{1}{4}\mathcal{R}g$.

Purely axial torsion waves

The irreducible pieces of torsion are

$$T^{(1)} = T - T^{(2)} - T^{(3)}, \quad T^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu, \quad T^{(3)} = *w,$$

where

$$v_\nu = \frac{1}{3} T^\lambda{}_{\lambda\nu}, \; w_\nu = \frac{1}{6}\sqrt{|\det g|}\, T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}.$$

The irreducible pieces $T^{(1)}$, $T^{(2)}$ i $T^{(3)}$ are called *tensor torsion*, *trace torsion*, and *axial torsion* respectively.

We are only interested in torsion which is *purely axial*, i.e. only $T^{(3)}$ is nonzero. It has been suggested that one can interpret the axial component of torsion as the Hodge dual of the electromagnetic vector potential.

The external field, or the perturbation is introduced into the model as a classical real differential geometric connection with torsion.

Connection

We work in Minkowski space $\mathbb{M}^4$, which is a real 4-manifold with a global coordinate system (x^0, x^1, x^2, x^3) and metric $g_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$.

The connection Γ we introduce separately and explicitly. The external field (perturbation) is introduced into the model as a classical real differential geometric connection with torsion

$$T = *A, \tag{1}$$

where A is the vector potential of the external electromagnetic field (given real valued vector function). We take the connection coefficients to be

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} A_\kappa \varepsilon^{\kappa\lambda}{}_{\mu\nu}, \tag{2}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$.

Note: The connection (2) is metric compatible with torsion (1), i.e. $\nabla g \equiv 0$.

Curvature

Metric compatibility implies there are only 6 possible nonzero pieces of curvature. These are given explicitly as:

$$\begin{aligned}
R^{(1)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{2} g_{\kappa[\nu} A_{\mu]} A_\lambda - \frac{1}{2} g_{\lambda[\nu} A_{\mu]} A_\kappa - \frac{1}{4} g_{\kappa[\nu} g_{\mu]\lambda} A_\eta A^\eta \\
R^{(2)}{}_{\kappa\lambda\mu\nu} &= -\frac{1}{4} g_{\kappa[\nu} g_{\mu]\lambda} A_\eta A^\eta \\
R^{(3)} &= 0 \\
R^{(4)}{}_{\kappa\lambda\mu\nu} &= -\frac{1}{4} \varepsilon_{\kappa\lambda\mu\nu} \partial_\eta A^\eta \\
R^{(5)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{2} \varepsilon^\eta{}_{\lambda\mu\nu} \partial_{(\kappa} A_{\eta)} - \frac{1}{2} \varepsilon^\eta{}_{\kappa\mu\nu} \partial_{(\lambda} A_{\eta)} - \frac{1}{4} \varepsilon_{\kappa\lambda\mu\nu} \partial_\eta A^\eta \\
R^{(6)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{2} g_{\kappa[\nu} (*dA)_{\mu]\lambda} - \frac{1}{2} g_{\lambda[\nu} (*dA)_{\mu]\kappa}
\end{aligned}$$

Only five non-zero curvature pieces. Note that the generalised Weyl curvature, i.e. the irreducible piece of curvature defined by the conditions $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}$, $\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0$, $Ric = 0$, is always identically equal to zero in purely axial torsion waves.

Discussion

Curvature also described using another formalism, but given completely explicitly.

There are clear applications to solutions of quadratic metric-affine gravity, i.e. pp-spaces (generalised pp-waves with purely axial torsion) with non-trivial metric. Also, this construction yields a class of Weitzenböck spaces ($R = 0$, $T \neq 0$).

Completely new solution of quadratic metric-affine gravity?

Can be viewed as a mathematical model for a massless particle? The spinor field which determines their complexified curvature satisfies the massless Dirac (or Weyl's) equation

$$\sigma^\mu{}_{ab} \nabla_\mu \xi^a - \frac{1}{2} T^\eta{}_{\eta\mu} \sigma^\mu{}_{ab} \xi^a = 0.$$

Now involved in further spectral analysis of the properties of the massless Dirac operator.