Physical interpretation of pp-waves with axial torsion

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We consider generalised pp-waves with purely axial torsion, which we previously showed to be new vacuum solutions of quadratic metric-affine gravity. Our analysis shows that classical pp-waves of parallel Ricci curvature should not be viewed on their own. They are a particular representation of a wider class of solutions, namely generalised pp-waves of parallel Ricci curvature. We compare our pp-waves with purely axial torsion to solutions of Einstein-Weyl theory, the classical model describing the interaction of gravitational and massless neutrino fields.

Keywords: Metric-affine gravity; Einstein–Weyl theory; PP-waves; Torsion waves

1. Introduction

Spacetime is considered to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. This approach, where the connection is viewed independently from the metric is called *metric-affine gravity*. In *quadratic* metric-affine gravity, we define the action as $S := \int q(R)$ where q is an $O(1,3)$–invariant quadratic form on curvature R. Independently varying the action with respect to the metric g and the connection Γ produces the system of Euler–Lagrange equations which we will write symbolically as

$$\partial S/\partial g = 0 \tag{1}$$

$$\partial S/\partial \Gamma = 0. \tag{2}$$

We consider a pp-wave as a Riemannian spacetime which admits a parallel spinor field. Classical pp-waves of parallel Ricci curvature were shown to be solutions of (1), (2) by Vassiliev.[1,2] In our previous paper Ref. 3, we introduced generalised pp-waves with purely axial torsion as metric compatible spacetimes with pp-metric and torsion $T := *A$, where A is a real vector field defined by $A = k(\varphi)l$, where l is a real parallel null lightlike vector and $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function of the phase $\varphi : M \mapsto \mathbb{R}$, $\varphi(x) := \int l \cdot dx$. If we were to write down the pp-metric locally as

$$ds^2 = 2\,dx^0\,dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1,x^2,x^3)\,(dx^3)^2,$$

in some local coordinates (x^0,x^1,x^2,x^3), for $l^\mu = (1,0,0,0)$ and $m^\mu = (0,1,\mp\mathrm{i},0)$ we get that $\varphi(x) = x^3 + \text{const}$. The torsion T is purely axial and the connection of a generalised pp-wave with purely axial torsion is metric compatible. We have shown that generalised pp-waves with purely axial torsion of parallel $\{Ric\}$ are solutions of (1), (2) in the Yang–Mills case. The remarkable property is that the curvature of a generalised pp-wave is a sum of the curvature of the underlying classical pp-space

$$-\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f \tag{3}$$

and the curvature

$$\frac{1}{4}k(\varphi)^2 \mathrm{Re}\,((l \wedge m) \otimes (l \wedge \overline{m})) \mp \frac{1}{2}k'(\varphi)\mathrm{Im}\,((l \wedge m) \otimes (l \wedge \overline{m})) \tag{4}$$

generated by a axial torsion wave traveling over the pp-space. Ricci curvature is

$$Ric = \frac{1}{2}\left(f_{11} + f_{22} - k^2\right)(l \otimes l), \tag{5}$$

where $f_{\alpha\beta} = \partial_\alpha \partial_\beta f$ and scalar curvature $\mathcal{R}$ is equal to zero. Similarly, the property that curvatures (3) and (4) add up was also present in the case of generalised pp-waves with purely tensor torsion, see Refs. 4, 5. In our previous paper Ref. 4, we gave the physical interpretation of generalised pp-waves with purely tensor torsion constructed in Ref. 5. Similarly to the approach of Ref. 4, now we want to compare the generalised pp-waves with purely axial torsion to the solutions of the classical models describing the interaction of gravitational and massless neutrino fields, namely Einstein–Weyl theory.

Our torsion and torsion generated curvature can be interpreted as waves traveling at speed of light. The underlying classical pp-space of parallel Ricci curvature can then be viewed as the gravitational imprint created by a wave of some massless matter field. As pointed out in Ref. 4, such a situation occurs in Einstein–Weyl theory. We choose to deal with the complexified curvature

$$\mathfrak{R} := r\,(l \wedge m) \otimes (l \wedge \overline{m}),$$

where $r := \frac{1}{4}k^2 - \frac{i}{2}k'$. Note that the function r is a function of the phase φ and the curvature (4) generated by the axial torsion is equal to $\mathrm{Re}(\mathfrak{R})$. The curvature $\mathfrak{R}$ is polarized, i.e. ${}^*\mathfrak{R} = \mathfrak{R}^* = \pm \mathrm{i}\mathfrak{R}$, and it can be written as

$$\mathfrak{R}_{\alpha\beta\gamma\delta} = \sigma_{\alpha\beta ab}\,\omega^{abcd}\,\overline{\sigma}_{\gamma\delta cd}, \tag{6}$$

where ω is some symmetric rank 4 spinor and $\sigma_{\alpha\beta}$ are second order Pauli matrices defined by $\sigma_{\alpha\beta ac} := \frac{1}{2}\big(\sigma_{\alpha a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\beta c\dot{d}} - \sigma_{\beta a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\alpha c\dot{d}}\big)$ and $\overline{\sigma}$ denotes their complex conjugation. Resolving (6) with respect to ω yields

$$\omega = \xi \otimes \xi \otimes \xi \otimes \xi, \tag{7}$$

where

$$\xi := r^{1/4}\,\chi \tag{8}$$

and $\chi^a = (1,\ 0)$ is the parallel spinor field of the underlying pp-space. Formula (7) shows that the rank 4 spinor ω is the 4th tensor power of a rank 1 spinor ξ. Hence, the curavture $\mathfrak{R}$ is completely determined by the rank 1 spinor field ξ. The spinor (8) satisfies the massless Dirac (or Weyl's) equation (13). Indeed, as $\nabla\chi = 0$, checking that ξ satisfies equation (13) reduces to checking that $(r^{1/4})'\,\sigma^\mu{}_{a\dot{b}}\,l_\mu\,\chi^a = 0$, which is straightforward using Pauli matrices for pp-waves from Ref. 4 and the explicit formulae for l and χ in local coordinates.

2. Einstein–Weyl Field Equations

We consider the action as

$$S_{EW} := 2i\int\left(\xi^a\,\sigma^\mu{}_{a\dot b}\,(\{\nabla\}_\mu\bar\xi^{\dot b})\;-\;(\{\nabla\}_\mu\xi^a)\,\sigma^\mu{}_{a\dot b}\,\bar\xi^{\dot b}\right)+K\int\mathcal{R}, \tag{9}$$

with the constant $K = c^4/16\pi G$. In Einstein–Weyl theory the connection is assumed to be Levi-Civita, so we obtain the Einstein–Weyl field equations varying the action (9) with respect to the metric and the spinor, i.e.

$$\partial S_{EW}/\partial g = 0, \tag{10}$$

$$\partial S_{EW}/\partial \xi = 0. \tag{11}$$

The massless Dirac equation is obtained by varying the action (9) with respect to the spinor. The variation of the first term of the action (9) with respect to the metric yields the energy momentum tensor. For the detailed derivation of formula for the energy momentum tensor see Appendix B of Ref. 4. The explicit representation of the Einstein–Weyl field equations (10), (11) is

$$\begin{aligned}&\frac{i}{2}\left[\sigma^\nu{}_{a\dot b}\left(\bar\xi^{\dot b}\{\nabla\}^\mu\xi^a-\xi^a\{\nabla\}^\mu\bar\xi^{\dot b}\right)+\sigma^\mu{}_{a\dot b}\left(\bar\xi^{\dot b}\{\nabla\}^\nu\xi^a-\xi^a\{\nabla\}^\nu\bar\xi^{\dot b}\right)\right]\\&+i\left(\xi^a\,\sigma^\eta{}_{a\dot b}\,(\{\nabla\}_\eta\bar\xi^{\dot b})g^{\mu\nu}-(\{\nabla\}_\eta\xi^a)\,\sigma^\eta{}_{a\dot b}\,\bar\xi^{\dot b}g^{\mu\nu}\right)-KRic^{\mu\nu}+\frac{K}{2}\mathcal{R}g^{\mu\nu}=0,\end{aligned} \tag{12}$$

$$\sigma^\mu{}_{a\dot b}\{\nabla\}_\mu\,\xi^a = 0. \tag{13}$$

3. Comparison of Metric-affine and Einstein–Weyl Solutions

The examination of the Eistein–Weyl field equations has a long history, see Refs. 6, 7, 8, 9, 10, 11, 12, 13, 14. One review of known solutions of Einstein–Weyl theory is given in Ref. 4. The nonlinear system of equations (12), (13) has solutions in the form of pp-waves. We wish to present a class of explicit solutions of (12), (13) where the metric g is in the form of the pp-metric and the spinor ξ as in (8). The spinor (8) satisfies the massless Dirac equation (13). The scalar curvature is zero and as the spinor χ appearing in formula (8) is parallel, hence the equation (12) now becomes

$$\frac{i}{2}\sigma^\nu{}_{a\dot b}\left(\bar\xi^{\dot b}\{\nabla\}^\mu\xi^a-\xi^a\{\nabla\}^\mu\bar\xi^{\dot b}\right)+\frac{i}{2}\sigma^\mu{}_{a\dot b}\left(\bar\xi^{\dot b}\{\nabla\}^\nu\xi^a-\xi^a\{\nabla\}^\nu\bar\xi^{\dot b}\right)-KRic^{\mu\nu}=0.$$

We now need to determine under which conditions the above equation is satisfied. Substituting formulae (5), (8) into the above equation, and using $\nabla\chi = 0$, we get that

$$i(\sigma^\nu{}_{a\dot b}l^\mu+\sigma^\mu{}_{a\dot b}l^\nu)\Big((r^{1/4})'\;\overline{r^{1/4}}-r^{1/4}\;(\overline{r^{1/4}})'\Big)\,\chi^a\bar\chi^{\dot b}=Kl^\mu l^\nu\big(f_{11}+f_{22}-k(x^3)^2\big)\,.$$

The condition that a pp-wave needs to satisfy to be a solution of Einstein–Weyl is

$$f_{11}+f_{22}=k(x^3)^2+\frac{2i}{K}\left((r^{1/4})'\;\overline{r^{1/4}}-r^{1/4}\;(\overline{r^{1/4}})'\right), \tag{14}$$

since $\sigma^{\mu}{}_{a\dot{b}}\chi^a\overline{\chi}^{\dot{b}} = l^\mu$. The complex valued function $r(\varphi)$ can be chosen arbitrarily and for the fixed function $k(\varphi)$ it uniquely determines the RHS of (14). The main difference between the two models is that in the metric-affine model the generalised pp-wave solutions have parallel $\{Ric\}$ curvature, whereas in the Einstein–Weyl model the pp-wave type solutions do not necessarily have parallel Ricci curvature. The comparison of this two types of solutions becomes much clearer in the case of the monochromatic solutions as was done in Ref. 4. In the metric-affine case the Laplacian of f can be any constant, while in the Einstein–Weyl case it is required for it to be a particular constant, which is the consequence of conformal invariance of the metric-affine model and the presence of the gravitational constant in the Einstein–Weyl. The generalised pp-waves of parallel Ricci curvature are very similar to pp-type solutions of the Einstein–Weyl model. According this conclusion, similarly to Ref. 4, we propose that generalised pp-waves with purely axial torsion and parallel Ricci curvature represent a metric-affine model for the massless neutrino.

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