Purely Axial Torsion Waves in Metric–affine Gravity†

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BMS Mathematical Conference
University of Sarajevo

12th July 2018

† Joint work with Elvis Baraković

Structure of talk

- Introduction and motivation
- Extended theories of gravity
- pp-waves with torsion
- Curvature characterisation of purely axial torsion waves

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Spacetime considered a connected real 4–manifold, equipped with a (Lorentzian) metric g (pseudo-Riemannian).

Einstein-Hilbert action

$$\frac{c^4}{16\pi G}\int \mathcal{R}.$$

Einstein's equations

$$\underbrace{Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}}_{GEOMETRY} = \underbrace{\frac{8\pi G}{c^4} T_{\mu\nu}}_{MATTER}.$$

Link between *spacetime* and matter which *bends* it.

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Consequences and tests of Einstein's GR

- Precession of the perihelion of Mercury;
- Deflection of light near the sun;
- Gravitational redshift;
- Temporal delay of radar;
- Global Positioning System (GPS);
- Black holes;
- Expansion of universe;
- Singularities;
- gravitational waves.

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Mechanical waves

- Rock thrown in water - transverse waves;
- Vibrations of vocal chords - longitudinal waves;
- Tsunami inducing earthquakes - surface waves.

Need material environment to propagate.

Particle position $u(\vec{r},t)$ at given time t satisfies homogenous *wave equation*

$$\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}+\frac{\partial}{\partial z^2}-\frac{1}{v^2}\frac{\partial}{\partial t^2}\right)u(\vec{r},t)=0.$$

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Electromagnetic waves

Equations of an electromagnetic field.

James Clark Maxwell 1861-62

$$\text{Gauss' Law of electricity:} \quad \oint\limits_{\substack{\text{closed}\\ \text{surface}}} \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$\text{Gauss' Law of magnetism:} \quad \oint\limits_{\substack{\text{closed}\\ \text{surface}}} \vec{B}\cdot d\vec{A} = 0$$

$$\text{Faraday's Law:} \quad \oint \vec{E}\cdot d\vec{s} = -\frac{d\phi_B}{dt}$$

$$\text{Ampere-Maxwell Law:} \quad \oint \vec{B}\cdot d\vec{s} = \mu_0\epsilon_0\frac{d\phi_E}{dt} + \mu_0 i_{enc}$$

Gravitational waves in GR

Mathematical proof of existence of gravitational waves in GR.

Take the following metric, far from the field source:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$$

where h is a small perturbation (wave) and $g^{(0)}$ is the metric of specetime where we observe the wave.

Einstein's equations become:

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{1}{c}\frac{\partial}{\partial t^2}\right) h_{\mu\nu} = 0.$$

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What is oscillating?

Mechanical wave - transfer of oscillations through material medium.

EM wave - transfer of electric and magnetic field of an accelerated charged particle.

Gravitational waves - the *metric* itself is oscillating! Distance between two infitesimal points in spacetime with metric g is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu .$$

If we take square root of RHS and integrate between points, we get arc length - distance between points in curved spacetime.

Metric oscillation means oscillation of distance between points.

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Extending GR

Developments in physics evoke possibility that treatment of spacetime might involve more than the Lorentzian spacetime of GR:

- Failure to quantize gravity;
- Generalisation of the 3D theory of elastic continua with microstructure to 4D spacetime of gravity;
- Description of hadron (or nuclear) matter in terms of extended structures;
- Study of early universe;
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Smallest departure from Lorentzian spacetime of GR:

- admit torsion - Riemann–Cartan spacetime;
- admit possible nonmetricity - *metric–affine* spacetime.

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Metric–affine gravity (MAG)

Spacetime a connected real 4–manifold $\mathcal{M}$ equipped with Lorentzian metric g and affine connection Γ, i.e.

$$\nabla_\mu u^\lambda = \partial_\mu u^\lambda + \Gamma^\lambda{}_{\mu\nu} u^\nu .$$

Independent linear connection Γ distinguishes MAG from GR from start.

Metric g and connection Γ viewed completely *independently*.

Unknowns of MAG : 10 independent metric components $g_{\mu\nu}$ and 64 connection $\Gamma^\lambda{}_{\mu\nu}$ coefficients.

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Quadratic metric-affine gravity (QMAG)

Action

$$S := \int q(R)$$

$q(R)$ is $O(1,3)$ invariant quadratic form on curvature R
16 R^2 terms and 16 coupling constants.

Action conformally invariant, unlike Einstein-Hilbert.

Why quadratic forms?

Yang-Mills action

$$q(R) := R^{\kappa}{}_{\lambda\mu\nu} R^{\lambda}{}_{\kappa}{}^{\mu\nu}.$$

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We vary action wrt g and Γ *independently*.

Euler-Lagrange system of equations

$$\frac{\partial S}{\partial g} = 0, \tag{1}$$

$$\frac{\partial S}{\partial \Gamma} = 0. \tag{2}$$

10+64 nonlinear PDEs with 10+64 unknowns.

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Generalising pp-waves

Classical pp-waves with parallel Ricci: Riemannian solution of QMAG.

Generalisation: connection not necessarily Levi-Civita.

We still use pp-metric but introduce explicitly given torsion.

- Pasic and Vassiliev:
  generalised pp-waves with purely *TENSOR* torsion;
- Barakovic and Pasic:
  generalised pp-waves with purely *AXIAL* torsion.

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Generalised pp-waves with tensor torsion of parallel Ricci curvature are solutions of (1) i (2).

Very simple explicit description.

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pp-waves with purely axial torsion

Definition

A generalised pp-wave with purely axial torsion is a metric compatible spacetime with pp-metric and torsion

$$T := *A$$

where A is a real vector field defined by $A = k(\varphi)\, l$, where $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function of the phase φ.

PP-waves with purely axial torsion

Curvature

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4}(k(x^3))^2 \operatorname{Re}\left((l \wedge m) \otimes (l \wedge \overline{m}) \right)$$
$$\mp \frac{1}{2} k'(x^3) \operatorname{Im}\left((l \wedge m) \otimes (l \wedge \overline{m}) \right)$$

Torsion

$$T = \mp \frac{i}{2} k \; l \wedge m \wedge \overline{m}.$$

Ricci curvature is $Ric = \frac{1}{2}(f_{11} + f_{22} - k^2) l \otimes l.$

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PP-WAVES WITH PURELY AXIAL TORSION

- Torsion of a generalised pp-wave is purely axial.
- Connection is metric compatible.
- Curvatures generated by the Levi-Civita connection and torsion simply add up to produce the formula for curvature.
- Weyl piece of torsion generated curvature is zero.
- Torsion also generates Ricci curvature.
- Curvature has three irreducible pieces.
- Scalar and pseudoscalar curvatures are zero.
- Ricci curvature is parallel if $f_{11} + f_{22} = (k(x^3))^2 + C$.
- Ricci curvature is zero if $f_{11} + f_{22} = (k(x^3))^2$.

The first result-published

Theorem

Generalised pp-waves with purely axial torsion of parallel $\{Ric\}$ are solutions of (1), (2) for the Yang-Mills action.

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Theorem

Generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of (1), (2) for the quadratic form with 11 R^2 terms.

Proof by "brute force"! Assumptions:

- Our spacetime is metric compatible.
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- Ricci curvature is symmetric.
- Scalar curvature $\mathcal{R}$ and pseudoscalar curvature $\mathcal{R}_*$ are zero.

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The second result

Explicit representation of the field equations:

$$0 = 2d_1 W^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} + d_2 \epsilon^{\eta\nu\alpha\beta} Ric_{\kappa\nu} Ric_*{}^{\kappa}{}_{\eta} - d_3 \epsilon^{\kappa\lambda\xi\alpha} W_{\kappa\lambda\mu}{}^{\beta} Ric_*{}^{\mu}{}_{\xi},$$

$$\begin{aligned}
0 = {} & d_1 \left\{ \nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + T_{\mu\eta\lambda} Ric_\kappa{}^\eta + T_{\mu\kappa\eta} Ric_\lambda{}^\eta \right\} \\
& - d_4 \left\{ (g_{\kappa\mu} \mathcal{W}^{\xi\zeta}{}_{\lambda\eta} - g_{\lambda\mu} \mathcal{W}^{\xi\zeta}{}_{\kappa\eta}) T^\eta{}_{\xi\zeta} + (g_{\kappa\mu} \epsilon^{\vartheta\zeta}{}_{\eta\lambda} - g_{\lambda\mu} \epsilon^{\vartheta\zeta}{}_{\eta\kappa}) T^\eta{}_{\xi\zeta} Ric_*{}^\xi{}_\vartheta \right\} \\
& + c_5 \left\{ \epsilon^{\eta\xi}{}_{\kappa\mu} \nabla_\xi Ric_{*\,\lambda\eta} - \epsilon^{\eta\xi}{}_{\lambda\mu} \nabla_\xi Ric_{*\,\kappa\eta} + \tfrac{1}{2} (\epsilon_\kappa{}^{\eta\xi\zeta} Ric_{*\,\lambda\eta} - \epsilon_\lambda{}^{\eta\xi\zeta} Ric_{*\,\kappa\eta}) T_{\mu\zeta\xi} \right\} \\
& - c_3 \Big\{ 2T^\eta{}_{\lambda\xi} \mathcal{W}^\xi{}_{\mu\kappa\eta} + 2T^\eta{}_{\xi\kappa} \mathcal{W}^\xi{}_{\mu\lambda\eta} + T_{\mu\xi\eta} \mathcal{W}_{\kappa\lambda}{}^{\eta\xi} + \epsilon^\vartheta{}_{\mu\eta\lambda} T^\eta{}_{\xi\kappa} Ric_*{}^\xi{}_\vartheta \\
& \qquad - \epsilon^\vartheta{}_{\mu\kappa\eta} T^\eta{}_{\xi\lambda} Ric_*{}^\xi{}_\vartheta - \epsilon^{\vartheta\xi}{}_{\eta\kappa} T^\eta{}_{\xi\lambda} Ric_{*\,\mu\vartheta} + \epsilon^{\vartheta\xi}{}_{\eta\lambda} T^\eta{}_{\xi\kappa} Ric_{*\,\mu\vartheta} \\
& \qquad + \epsilon^\vartheta{}_{\mu\lambda\kappa} \nabla_\xi Ric_*{}^\xi{}_\vartheta - \epsilon^{\vartheta\xi}{}_{\lambda\kappa} \nabla_\xi Ric_{*\,\mu\vartheta} \Big\}.
\end{aligned}$$

Physical interpretation

Conjecture

Generalised pp-waves of parallel Ricci curvature represent a metric-affine model for the massless neutrino.

- Elvis Barakovic, Vedad Pasic: *"Physical Interpretation of PP-waves with Axial Torsion"*, 14th Marcel Grossmann Meeting on General Relativity, Rome. Proceedings, World Scientific, 2017.
- Vedad Pasic, Elvis Barakovic: *"Axial Torsion Waves in Metric-affine Gravity"*, 14th Marcel Grossmann Meeting on General Relativity, Rome. Proceedings, World Scientific, 2017.

Purely axial torsion waves

Minkowski space $\mathbb{M}^4$ is a real 4-manifold with global coordinate system (x^0, x^1, x^2, x^3) and metric $g_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$.

Specify the manifold and the metric, but the connection Γ we introduce separately and explicitly.

The external field (perturbation) is introduced into the model as a classical real differential geometric connection with torsion

$$T = *A, \tag{3}$$

where A is vector potential of the external electromagnetic field (given real valued vector function).

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Connection - explicit

We equip our Minkowski 4-space with a non-trivial connection. We take the connection coefficients to be

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} A_{\kappa} \varepsilon^{\kappa\lambda}{}_{\mu\nu}, \tag{4}$$

Lemma

The connection (4) is metric compatible with torsion (3).

No matter what the torsion is, as long as it is purely axial, we have metric compatibility!

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Irreducible pieces of curvature

The Riemann tensor in general has only one antisymmetry, namely

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}. \tag{5}$$

We denote by $\mathbf{R}$ the 96-dimensional vector space of real rank 4 tensors which satisfy the condition (5) and we have the natural inner product on $\mathbf{R}$ given by

$$(R, Q) := R^{\kappa}{}_{\lambda\mu\nu} Q^{\lambda}{}_{\kappa}{}^{\mu\nu}. \tag{6}$$

We have an orthogonal decomposition $\mathbf{R} = \mathbf{R}^{+} \oplus \mathbf{R}^{-}$, where

$$\mathbf{R}^{\pm} = \{\mathbf{R} \in \mathbf{R} | \mathbf{R}_{\kappa\lambda\mu\nu} = \pm \mathbf{R}_{\mu\nu\kappa\lambda}\}.$$

Respective dimensions of these subspaces are 60 and 36. We are only interested in the subspace $\mathbf{R}^{-}$ as this is the vector space of curvatures generated by metric compatible connections.

Metric compatible irreducible decomposition

$$
\begin{aligned}
R^{(1)} &= \frac{1}{2}(g_{\kappa\mu}\overline{\mathcal{R}ic}_{\lambda\nu} - g_{\lambda\mu}\overline{\mathcal{R}ic}_{\kappa\nu} - g_{\kappa\nu}\overline{\mathcal{R}ic}_{\lambda\mu} + g_{\lambda\nu}\overline{\mathcal{R}ic}_{\kappa\mu}) \\
R^{(2)} &= \frac{1}{12}(g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu})\mathcal{R} \\
R^{(3)} &= \overline{R} - R^{(1)} - R^{(2)} - R^{(4)} \\
R^{(4)} &= -\frac{1}{24}\sqrt{|\det g|}\,\varepsilon_{\kappa\lambda\mu\nu}\breve{\mathcal{R}} \\
R^{(5)} &= \widehat{R} - R^{(6)} \\
R^{(6)} &= \frac{1}{2}(g_{\kappa\mu}\widehat{Ric}_{\lambda\nu} - g_{\lambda\mu}\widehat{Ric}_{\kappa\nu} - g_{\kappa\nu}\widehat{Ric}_{\lambda\mu} + g_{\lambda\nu}\widehat{Ric}_{\kappa\mu},
\end{aligned}
$$

where

$$\overline{R}_{\kappa\lambda\mu\nu} = \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} + R_{\mu\nu\kappa\lambda} - R_{\nu\mu\kappa\lambda})$$

$$\widehat{R}_{\kappa\lambda\mu\nu} = \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} - R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda})$$

$$\overline{Ric}_{\lambda\nu} = \overline{R}^{\kappa}{}_{\lambda\kappa\nu}, \quad \mathcal{R} = \overline{Ric}^{\lambda}{}_{\lambda} = R^{\kappa\lambda}{}_{\kappa\lambda},$$

$$\overline{\mathcal{R}ic}_{\lambda\nu} = \overline{Ric}_{\lambda\nu} - \frac{1}{4}g_{\lambda\nu}\mathcal{R}, \quad \widehat{Ric} = \widehat{R}^{\kappa}{}_{\lambda\kappa\nu},$$

$$\breve{\mathcal{R}} = \sqrt{|\det g|}\varepsilon^{\kappa\lambda\mu\nu}, \quad \overline{R}_{\kappa\lambda\mu\nu} = \sqrt{|\det g|}\varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}.$$

Irreducible pieces of curvature - axial

Theorem

The irreducible pieces of curvature generated by a purely axial torsion wave in Minkowski space are given by

$$R^{(1)}{}_{\kappa\lambda\mu\nu} = \frac{1}{2} g_{\kappa[\nu} A_{\mu]} A_\lambda - \frac{1}{2} g_{\lambda[\nu} A_{\mu]} A_\kappa - \frac{1}{4} g_{\kappa[\nu} g_{\mu]\lambda} A_\eta A^\eta$$

$$R^{(2)}{}_{\kappa\lambda\mu\nu} = -\frac{1}{4} g_{\kappa[\nu} g_{\mu]\lambda} A_\eta A^\eta$$

$$R^{(3)} = 0$$

$$R^{(4)}{}_{\kappa\lambda\mu\nu} = -\frac{1}{4} \varepsilon_{\kappa\lambda\mu\nu} \partial_\eta A^\eta$$

$$R^{(5)}{}_{\kappa\lambda\mu\nu} = \frac{1}{2} \varepsilon^\eta{}_{\lambda\mu\nu} \partial_{(\kappa} A_{\eta)} - \frac{1}{2} \varepsilon^\eta{}_{\kappa\mu\nu} \partial_{(\lambda} A_{\eta)} - \frac{1}{4} \varepsilon_{\kappa\lambda\mu\nu} \partial_\eta A^\eta$$

$$R^{(6)}{}_{\kappa\lambda\mu\nu} = \frac{1}{2} g_{\kappa[\nu} (*dA)_{\mu]\lambda} - \frac{1}{2} g_{\lambda[\nu} (*dA)_{\mu]\kappa}$$

Discussion

Only five non-zero curvature pieces.

Generalised Weyl curvature, i.e. the irreducible piece of curvature defined by the conditions

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda},\ \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0,\ Ric = 0,$$

is always identically equal to zero.

Clear applications to solutions of quadratic metric-affine gravity, i.e. pp-spaces. This construction yields a class of Weitzenböck spaces ($R = 0, T \neq 0$).

Curvature also described using another formalism, but given completely explicitly.

Thank You and welcome to Tuzla!