New Vacuum Solutions for Quadratic Metric–affine Gravity

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Structure of the thesis

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- Introduction

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- PP-waves with torsion

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- New vacuum solutions for quadratic metric-affine gravity

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- Appendices

Metric–affine gravity

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- Spacetime considered to be a connected real 4–manifold *M* equipped with a Lorentzian metric *g* and an affine connection Γ, i.e.

$$\nabla_\mu u^\lambda = \partial_\mu u^\lambda + \Gamma^\lambda{}_{\mu\nu} u^\nu .$$

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- 10 independent components of $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are the unknowns of MAG
- **Definition.** We call a spacetime $\{M, g, \Gamma\}$ *Riemannian* if the connection is Levi–Civita (i.e. $\Gamma^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$), and *non-Riemannian* otherwise.

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where $q(R)$ is a Lorentz invariant purely quadratic form on curvature.
- The quadratic form $q(R)$ has 16 R^2 terms with 16 real coupling constants.
- Action conformally invariant, unlike Einstein–Hilbert.

Quadratic metric–affine gravity

- Independent variation of g and Γ produces the system of Euler–Lagrange equations

$$\partial S/\partial g = 0, \tag{1}$$
$$\partial S/\partial \Gamma = 0. \tag{2}$$

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- Einstein spaces (Yang, Mielke);
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- Certain explicitly given torsion waves (Singh and Griffiths);
- Triplet ansatz (Hehl, Macías, Obukhov, Esser, ...);
- Minimal pseudoinstanton generalisation (Obukhov).

Classical pp-waves

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$$\mathrm{d}s^2 = 2\,\mathrm{d}x^0\,\mathrm{d}x^3 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 + f(x^1, x^2, x^3)\,(\mathrm{d}x^3)^2$$

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  in some local coordinates.
- Well known spacetimes in GR, simple formula for curvature - only trace free Ricci and Weyl pieces.

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- **Definition** A *generalised pp-wave* is a metric compatible spacetime with pp-metric and torsion

$$T := \frac{1}{2}\mathrm{Re}(A \otimes dA).$$

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- Curvature of a generalised pp-wave is

$$R=-\frac{1}{2}(l\wedge\{\nabla\})\otimes(l\wedge\{\nabla\})f+\frac{1}{4}\mathrm{Re}\left((h^2)''\,(l\wedge m)\otimes(l\wedge m)\right).$$

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- Torsion of a generalised pp-wave is

$$T = \mathrm{Re}\left((a\,l + b\,m) \otimes (l \wedge m)\right),$$

where

$$a := \frac{1}{2}h'(\varphi)\,k(\varphi), \quad b := \frac{1}{2}h'(\varphi)\,h(\varphi).$$

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- **Theorem** Generalised pp-waves of parallel Ricci curvature are solutions of the field equations (1) and (2).
- In special local coordinates, ‘parallel Ricci curvature’ is written as $f_{11} + f_{22} = \mathrm{const}$.
- Generalised pp-waves of parallel Ricci curvature admit a simple explicit description.

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- Proof by 'brute force'.
- We write down the field equations (1) and (2) for general metric compatible spacetimes and substitute the formulae for torsion, Ricci and Weyl into these.
- Together with $\nabla Ric = 0$, we get the result.
- This result was first presented in :
  "PP-waves with torsion and metric affine gravity", 2005 V. Pasic, D. Vassiliev, *Class. Quantum Grav. 22 3961-3975*.

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- Torsion and torsion generated curvature are waves traveling at the speed of light.
- Underlying pp-space can be viewed as the ‘gravitational imprint’ created by wave of some massless field.
- Mathematical model for neutrino?

Metric-affine model for a neutrino

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- Neutrino field in metric compatible spacetime described by

$$S_{\text{neutrino}} := 2i \int \left(\xi^a \, \sigma^\mu{}_{a\dot{b}} \, (\nabla_\mu \bar{\xi}^{\dot{b}}) \; - \; (\nabla_\mu \xi^a) \, \sigma^\mu{}_{a\dot{b}} \, \bar{\xi}^{\dot{b}} \right),$$

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- Constructed pp-wave type solutions of Einstein-Weyl model

$$S_{\mathrm{EW}} := k \int \mathcal{R} + S_{\mathrm{neutrino}},$$

$$\partial S_{\mathrm{EW}} / \partial g = 0,$$

$$\partial S_{\mathrm{EW}} / \partial \xi = 0.$$